Math 417 Problem Set 6

Starred (*) problems are due Friday, March 4.

43. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$, thought of as a function frm the group $(\mathbb{R}, +, 0)$ to the group $(\mathbb{R}^+, *, 1)$ of positive real numbers under multiplication, is an isomorphism of groups.

44. Show that if G_1, G_2 are groups, $H_1 \leq G_1$ is a subgroup of G_1, and $\varphi : G_1 \to G_2$ is a homomorphism, then $H_2 = \{\varphi(h) \ : \ h \in H_1\}$ (the *image* of H_1) is a subgroup of G_2.

(*) 45. (Gallian, p.220, # 10) Suppose that G is a *dihedral group* (i.e., a group of symmetries of some regular n-gon), and define the function $\varphi : G \to H = (\{-1, 1\}, *, 1)$ to the group H (isomorphic to $\mathbb{Z}_2$) by $\varphi(\text{rotation}) = 1$ and $\varphi(\text{reflection}) = -1$. Show that φ is a homomorphism.

[Hint: Problem Set # 1 !]

46. (Gallian, p.139, # 26) Show that the function $\varphi : \mathbb{Z}_{16}^* \to \mathbb{Z}_{16}^*$ given by $\varphi(a) = a^3$ is an isomorphism. What about $a \mapsto a^5$? Or $a \mapsto a^7$?

(*) 47. (Gallian, p.140, # 30) Suppose that $\varphi : (\mathbb{Z}_{50}, +, 0) \to (\mathbb{Z}_{50}, +, 0)$ is an isomorphism and $\varphi(11) = 13$. Show that, for all x, $\varphi(x) = kx$ for a certain k, and find k !

(*) 48. (Gallian, p.141, #42) Suppose that G is a finite *abelian* group, and that no element of G has order 2. Show that the function $\varphi : G \to G$ given by $\varphi(g) = g^2$ is an isomorphism. Show that if G is infinite then φ is a homomorphism, but need not be an isomorphism.

[Hint: show that the hypothesis about orders implies that φ is injective.]

49. (Gallian, p.141, # 22 (sort of)) Show that if $\varphi : G \to G$ is an homomorphism from G to itself, then $H = \{g \in G \ : \ \varphi(g) = g\}$ is a subgroup of G.

[N.B.: H is called the *fixed subgroup* of φ .]

50. (Gallian, p.223, # 55 (part)) Let $(\mathbb{Z}[x], +, 0)$ be the group of polynomials with integer coefficients, under addition, and let $k \in \mathbb{Z}$. Show that the function $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ given by $\varphi(p(x)) = p(k)$ [the 'evaluation function'] is a homomomorphism.