Math 417 Second Exam

Guidelines: The following problems constitute the second exam, which will take place Monday, May 2, from 10:00-12:00noon, in Burnett 205. You may (and probably must!) prepare solutions to the problems in advance of the exam. You may turn in these prepared solutions at any time prior to the exam time, in which case you need not attend the actual exam. In preparing your solutions, you may consult our text [Gallian's *Contemporary Abstract Algebra* (any edition)], your class notes, any papers/solutions handed out in (our...) class, and your (Math 417) instructor. No other sources may be consulted, and if you run into any information in the normal course of your daily life which appears to be relevant to the solution of any of the problems below, you should avoid following that information, to the greatest extent that your other obligations allow. Each problem letter (A,B,C,D,E) is worth equal credit.

A. If $(G, \cdot, e)$ is a group, then the *opposite group* $(G^{op}, *, e)$ is the group with the same elements G, but with multiplication $g * h = h \cdot g$.

A.1. Show that G^{op} is in fact a group, and show that it is isomorphic to G . [Hint: what group operation reverses the order in a product?]

A.2. Show that if $H \leq G$ is a subgroup of G, then H is also a subgroup of G^{op}.

B.1. If G is a group, $H \leq G$ is a subgroup, and $a, b \in G$, show that $aH = bH \Leftrightarrow Ha^{-1} = Hb^{-1}$.

B.2. Show that the symmetric group S_3 contains a subgroup H and elements $a, b \in S_3$ so that $aH = bH$ but $Ha \neq Hb$.

C. A group G is called *perfect* if every element $g \in G$ can be expressed as a product of commutators $\alpha\beta\alpha^{-1}\beta^{-1}$, i.e., $g = (\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}) \cdots (\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1})$ for some $\alpha_i, \beta_i \in G$.

C.1. Show that, if $n \geq 5$, every 3-cycle (a, b, c) in the alternating group A_n can be expressed as a commutator of 3-cycles, $(a, b, c) = (a_1, b_1, c_1)(a_2, b_2, c_2)(a_1, b_1, c_1)^{-1}(a_2, b_2, c_2)^{-1}$.

C.2. Show, using part 1, that for every $n \geq 5$, the alternating group A_n is perfect.

[N.B.: A_3 and A_4 are, on the other hand, not perfect: A_3 is abelian (and they are never perfect), and the result of C.1. is false for A_4 ...]

D. Let G be a finite group, $N \triangleleft G$ a normal subgroup of G, and $H \leq G$ a subgroup of G. Suppose that $|H|$ and $|N|$ are relatively prime (ie., $\gcd(|H|, |N|) = 1$). Show that the quotient group G/N contains a subgroup isomorphic to H.

[Hint: show that there is an injective homomomrpism $\varphi : H \to G/N$.]

E. Show that if G is a group of order $741 = 3 \cdot 13 \cdot 19$, then G contains at least two normal p-Sylow subgroups.