Symmetries fix the center of mass

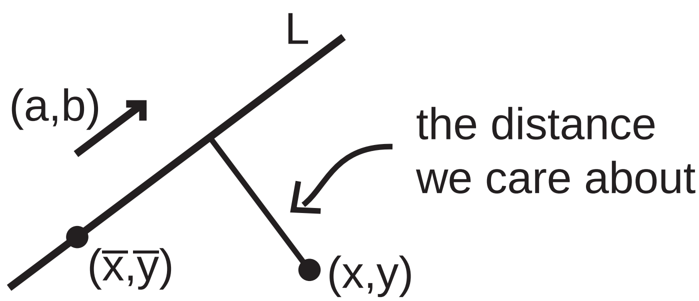
Central (sorry...) to our study of symmetry is the fact that any rigid motion R of the plane (although this can be extended to higher dimensions) which carries a region Q to itself, $R(Q) = Q$, must satisfy $R(\overline{x}, \overline{y}) = (\overline{x}, \overline{y})$, where $(\overline{x}, \overline{y})$ is the center of mass (or, more properly, centroid) of the region Q.

This follows from a bit of multivariable calculus and linear algebra. Recall (!) that the coordinates $(\overline{x}, \overline{y})$ of the centroid are those values for which

$$\int\int_Q x - \overline{x}\, dA = 0 \text{ and } \int\int_Q y - \overline{y}\, dA = 0$$

[The idea: Q will 'balance' on the vertical and horizontal lines through the centroid. $x - \overline{x}$ and $y - \overline{y}$ are the 'signed' distances from points in Q to these lines.]

This in turn implies that Q will balance on any line through the centroid. Seeing this amounts to working out what the signed distance is from a point in Q to a line passing through the centroid, and noting that the integral of that distance, over Q, is also 0. Such a line can be expressed, parametrically, as $L(t) = (\overline{x}, \overline{y}) + t(a, b)$, where, for convenience, we will assume that $a^2 + b^2 = 1$ (i.e., our direction vector has length 1). Finding the signed distance we need amounts to writing the vector $\vec{v} = (x - \overline{x}, y - \overline{y})$ as a linear combination of the (orthonormal) vectors $\vec{w}_1 = (a, b)$ and $\vec{w}_2 = (b, -a)$, and taking the $(b, -a)$-coordinate (see figure).



But this can be done by multiplying $\vec{v}$ by the inverse of the matrix with columns $\vec{w}_1, \vec{w}_2$, which, since these vectors are orthonormal, is the transpose. So, computing:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}^T = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \text{ and } \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix} = \begin{pmatrix} a[x - \overline{x}] + b[y - \overline{y}] \\ b[x - \overline{x}] - a[y - \overline{y}] \end{pmatrix}.$$

So (check that!) $\begin{pmatrix} x - \overline{x} \\ y - \overline{y} \end{pmatrix} = (a[x - \overline{x}] + b[y - \overline{y}]) \begin{pmatrix} a \\ b \end{pmatrix} + (b[x - \overline{x}] - a[y - \overline{y}]) \begin{pmatrix} b \\ -a \end{pmatrix}$,

making $b[x - \overline{x}] - a[y - \overline{y}]$ the signed distance from (x, y) to the line. But then

$\int\int_Q b[x - \overline{x}] - a[y - \overline{y}\, dA = b \int\int_Q x - \overline{x}\, dA - a \int\int_Q y - \overline{y}\, dA = b(0) - a(0) = 0$, as desired.

This enables us to establish our main result. Any rigid motion R carries lines to lines, and so it carries a line L through the centroid of Q to some other line L'. But because R is an isometry, it preserves distance, and so the distance from a point $(x, y) \in Q$ to L will be equal to the distance from $R(x, y)$ to L'. [Ah, this is because an isometry will also preserve angle; this is because the Law of Cosines will let you compute (the cosine of) an angle using length measurements alone: $c^2 = a^2 + b^2 - 2ab\cos(C)$ let's you compute C. Or: under an isometry the lengths in a triangle won't change, so the angles won't, either. So the orthogonal projection of a point will be taken to the orthogonal projection.] [In addition, R will either preserve all of the signs ("orientation-preserving") or reverse all of the signs ("orientation-reversing").] This in turn means that if you integrate the distance to L' over the region Q, you will still get 0; it is the same as integrating the distance to L, by thinking of the isometry R as a change of variables function taking the region Q to the region Q (!).

But this means that L' is a line through the centroid! Some parallel line L'' must pass through the centroid, but our integral for L'' will differ from the integral for L' by the distance between the lines times the area of Q (since at every point in Q the two integrands will differ by the distance betweeen the lines). The only way for the integral for L'' to give 0, therefore, is to have $L' = L''$. So our symmetry R must carry a line through the centroid of Q to a line through the centroid.

But! Now take two lines through the centroid (which determine the centroid!) L_1, L_2; so $L_1 \cap L_2 = (\overline{x}, \overline{y})$. Then $R(L_1)$ and $R(L_2)$ are both lines through the centroid, so $\{(\overline{x}, \overline{y})\} \subseteq R(L_1) \cap R(L_2) = R(L_1 \cap L_2) = R(\overline{x}, \overline{y})$. So $R(\overline{x}, \overline{y}) = (\overline{x}, \overline{y})$, and so R fixes the centroid of Q.

So by employing double integrals to compute centroids, the (orthogonal) projection from a point to a line, the change of basis formula from linear algebra, the Law of Cosines, and a change of variables for double integrals, we achieve our result. This all generalizes to higher dimensions; lines (and distances to them) are replaced, in $\mathbb{R}^n$, with $(n-1)$-dimensional subspaces (and distances to them), and you then need to use n 'hyperplanes' to determine the centroid, but most everything else stays the same...