Math 325

Continuous and one-to-one implies increasing or descreasing

Here is a <u>better</u> argument to establish the statement in the title: if I is an interval and $f : I \to \mathbb{R}$ is continuous and one-to-one, then f is either increasing or decreasing. This argument is essentially lifted from a (better!) textbook (than our own!), called *Calculus* by Michael Spivak. The idea is to use the Intermediate Value Theorem in a more 'unifed' way to show that if for one pair of points $x_0, y_0 \in I$ with $x_0 < y_0$ we have $f(x_0) < f(y_0)$, then for <u>every</u> pair of points $x_1, y_1 \in I$ with $x_1 < y_1$ we have $f(x_1) < f(y_1)$, <u>so</u> f is increasing. Note that, if <u>no</u> such pair of points $x_0, y_0 \in I$ exist, then we instead have the for every $x, y \in I$ with $x < y$ we have $f(x) > f(y)$ (since they can't be equal); that is, f is <u>decreasing</u>!

So suppose that $x_0, y_0 \in I$ with $x_0 < y_0$ and $f(x_0) < f(y_0)$, and suppose that $x_1, y_1 \in I$ with $x_1 < y_1$. What we will do is, simultaneously, walk from x_0 to x_1, while 'someone else' walks from y_0 to y_1, and compare the values of f at the pairs of points we are encountering. In symbols, we look at

$$\alpha(t) = (1-t)x_0 + tx_1 \text{ and } \beta(t) = (1-t)y_0 + ty_1, \text{ for } 0 \le t \le 1$$

Note that $\alpha, \beta : [0,1] \to \mathbb{R}$ are both linear functions (of the variable t), and therefore continuous. We also have $\alpha(0) = x_0$, $\alpha(1) = x_1$, $\beta(0) = y_0$, and $\beta(1) = y_1$, and, perhaps more importantly,

$$\beta(t) - \alpha(t) = (1-t)(y_0 - x_0) + t(y_1 - x_1)$$

is a sum of <u>non</u>-<u>negative</u> numbers (since $0 \le t \le 1$ and $y_0 - x_0, y_1 - x_1 > 0$) at least one of which is positive (since either $1 - t > 0$ (i.e., $t < 1$) or $t > 0$), so $\alpha(t) < \beta(t)$ for every $t \in [0,1]$, and so $f(\alpha(t)) \neq f(\beta(t))$ for every t, since f is one-to-one.

But! then the function $g : [0,1] \to \mathbb{R}$ given by $f(\beta(t)) - f(\alpha(t))$ is a difference of compositions of contiuous functions, so is continuous. Our analysis above says that g never takes the value 0, since $\beta(t) \neq \alpha(t)$ for every $t \in [0,1]$. Therefore, by the Intermediate Value Theorem, it <u>cannot</u> take both positive <u>and</u> negative values. So since

$$g(0) = f(\beta(0)) - f(\alpha(0)) = f(y_0) - f(x_0) > 0 \ ,$$

we must also have $g(1) = f(\beta(1)) - f(\alpha(1)) = f(y_1) - f(x_1) > 0$; that is, $f(x_1) < f(y_1)$.

So if for one pair of points $x_0 < y_0$ we have $f(x_0) < f(y_0)$, then for <u>every</u> pair of points $x_1 < y_1$ we have $f(x_1) < f(y_1)$, i.e., f is an increasing function. It then also follows that if for one pair of points $x_0 < y_0$ we have $f(x_0) > f(y_0)$, then for <u>every</u> pair of points $x_1 < y_1$ we have $f(x_1) > f(y_1)$, i.e., f is an decreasing function.

So every one-to-one continuous function defined on an interval is either increasing (if one pair of values indicates that it is increasing) or decreasing (if one pair of points indicates that it is decreasing).