Math 107H

Topics for the second exam

Technically, everything for the first exam! Plus:

Improper integrals

Fund Thm of Calc: $\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; f blows up at a or b or somewhere in between
integral is"improper"; usual technique doesn't work. Solution to this:

$\int_a^\infty f(x)\ \mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\ \mathrm{d}x$ $\qquad$ $\int_{-\infty}^b f(x)\ \mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\ \mathrm{d}x$

(blow up at a) $\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to a^+} \int_r^b f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)\ \mathrm{d}x$

(similarly for blowup at b (or both!))

$\int_a^b f(x)\ \mathrm{d}x = \lim_{s\to b^-} \int_a^s f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\ \mathrm{d}x$

(blows up at c (b/w a and b)) $\int_a^b f(x)\ \mathrm{d}x = \lim_{r\to c^-} \int_a^r f(x)\ \mathrm{d}x + \lim_{s\to c^+} \int_s^b f(x)\ \mathrm{d}x$

The integral <u>converges</u> if (all of the) limit(s) are finite

Comparison: $0 \le f(x) \le g(x)$ for all x;

if $\int_a^\infty g(x)\ \mathrm{d}x$ *converges*, so does $\int_a^\infty f(x)\ \mathrm{d}x$

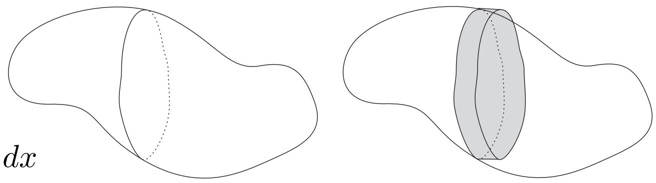
if $\int_a^\infty f(x)\ \mathrm{d}x$ *diverges*, so does $\int_a^\infty g(x)\ \mathrm{d}x$

Applications of integration

Volume by slicing. To calculate volume, aprroximate region by objects whose volume we <u>can</u> calculate.

Volume $\approx \sum$(volumes of 'cylinders')
$= \sum$(area of base)(height)
$= \sum$(area of cross-section)Δx_i .

So volume $= \int_{left}^{right}$ (area of cross section) dx

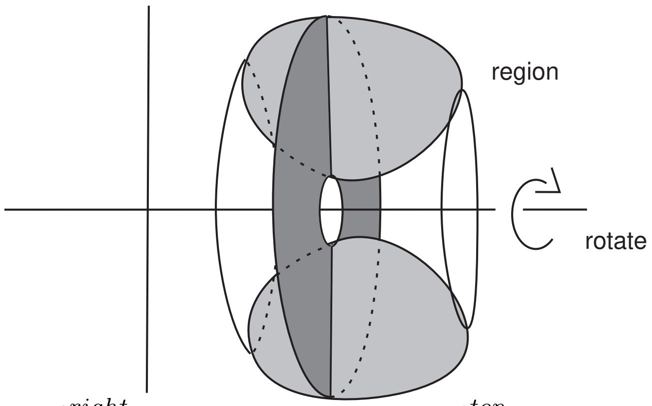


Solids of revolution: disks and washers. Solid of revolution: take a region in the plane and revolve it around an axis in the plane.

take cross-sections perpendicular to axis of revolution;
cross-section = disk (area=πr^2)
or washer (area=$\pi R^2 - \pi r^2$)

rotate around x-axis: write r (and R) as functions of x, integrate dx

rotate around y-axis: write r (and R) as functions of y, integrate dy

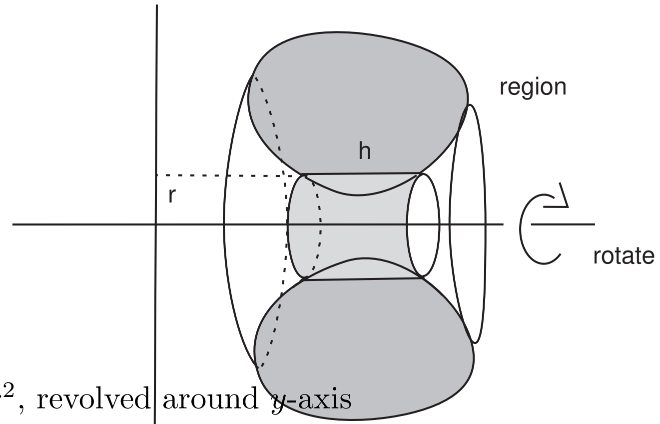


Otherwise, everything is as before: volume = $\displaystyle\int_{left}^{right} A(x)\, dx$ or volume = $\displaystyle\int_{bottom}^{top} A(y)\, dy$

The same is true if axis is <u>parallel</u> to $x-$ or $y-$axis; r and R just change (we add a constant).

Cylindrical shells. Different picture, same volume! Solid of revolution; use cylinders centered on the axis of revolution. The intersection is a cylinder, with area = (circumference)(height) = $2\pi rh$

volume = $\displaystyle\int_{left}^{right}$ (area of cylinder) dx or $\displaystyle\int_{bottom}^{top}$ (area of cylinder) dy!)



revolve around vertical line: integrate dx

revolve around horizontal line: integrate dy

Ex: region in plane between $y = 4x$, $y = x^2$, revolved around y-axis

left=0, right=4, $r = x$, $h = (4x - x^2)$ volume = $\displaystyle\int_0^4 2\pi x(4x - x^2)\, dx$

Arclength. Idea: approximate a curve by lots of short line segments; length of curve $\approx$ sum of lengths of line segments.

Line segment between $(c_i, f(c_i))$ and $(c_{i+1}, f(c_{i+1}))$:

$$\sqrt{1 + \left(\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}\right)^2} \cdot (c_{i+1} - c_i) \approx \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

<u>So</u> length of curve = $\displaystyle\int_{left}^{right} \sqrt{1 + (f'(x))^2}\, dx$

The problem: integrating $\sqrt{1 + (f'(x))^2}$! Sometimes, $1 + (f'(x))^2$ turns out to be a perfect square.....

Exponential growth and decay

In many situations, the rate of change of some quantity depends in a known way on the values of the quantity. A basic example is *radioactive decay*: if $f(t)$ is the amount of isotope at time t, then $f'(t) = kf(t)$ for some constant k (which depends upon the isotope). Such equation is called a *differential equation*, since it involves an (unknown) function as well as its derivative.

The equation for radiactive decay is one of a class of equations called *separable* equations. A differential equation is separable if it can be written as $y' = A(t)B(y)$

This allows us to 'separate the variables' and integrate with respect to dy and dt to get a solution:

$$\frac{1}{B(y)}dy = A(t)\ dt \text{ ; integrate both sides}$$

In the end, our solutions look like $F(y) = G(t) + c$, so it defines y *implicitly* as a function of t , rather than explicitly. In some cases we can invert F to get an explicit solution, but often we cannot.

For example, the separable equation $y' = ty^2$, $y(1) = 2$ has solution $\displaystyle\int \frac{dy}{y^2} = \int t\ dt + c$

so solving the integrals we get $(-1/y) = (t^2/2) + c$, or $y = -2/(t^2 + 2c)$; setting $y = 2$ when $t = 1$ gives $c = -1$.

Applying this approach to a radioactive decay problem, $y' = ky$, yields $y(t) = Ce^{kt}$, where the constant of integration C can be determined by setting $t = 0$; $y_0 = y(0) = Ce^0 = C$. So $y(t) = y_0e^kt$. The constant k can then be determined if we know the value of $y(t)$ for any other time t_0; $k = \dfrac{1}{t_0}\ln[y(t_0)/y_0]$.

Newton's Law of Cooling: This states that the rate of change of the temperature $T(t)$ of an object is proportional to the difference between its temperature and the ambient temperature of the air around it. The constant of proportionality depends upon the particular object (and the medium, e.g., air or water) it is in. In other words,

$$T' = k(A - T)$$

Since a cold object will warm up, and a warm object will cool down, this means that the constant k should be positive. This equation is separable, and we can find the solution

$$T(t) = A + (T(0) - A)e^{-kt}$$

Typically, k is not given, but can be determined by knowing the temperature at some other time t_1, by plugging into the equation above and solving for k.

Infinite sequences and series

Limits of sequences of numbers

A sequence is: a string of numbers; a function f:$\mathbf{N}\to\mathbf{R}$; write $f(n) = a_n$

a_n = n-th term of the sequence

Basic question: convergence/divergence

$\lim\limits_{n\to\infty} a_n = L$ (or $a_n \to L$) if

eventually all of the a_n are always as close to L as we like, i.e.

for any $\epsilon > 0$, there is an N so that if $n \geq N$ then $|a_n - L| < \epsilon$

Ex.: $a_n = 1/n$ converges to 0 ; can always choose N=$1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a single number

If a_n is increasing ($a_{n+1} \geq a_n$ for every n) and bounded from above

($a_n \leq M$ for every n, for some M) , then a_n converges (but not necessarily to M !)

limit is smallest number bigger than all of the terms of the sequence

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions

If $a_n \to L$ and $b_n \to M$, then

$(a_n + b_n \to L + M \qquad (a_n - b_n) \to L - M \qquad (a_n b_n) \to LM$, and

$(a_n/b_n) \to L/M$ (provided M, all b_n are $\neq 0$)

Sqeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all n large enough) and

$a_n \to L$ and $c_n \to L$, then $b_n \to L$

If $a_n \to L$ and f:$\mathbf{R}$→$\mathbf{R}$ is continuous at L, then $f(a_n) \to f(L)$

if $a_n = f(n)$ for some function f:$\mathbf{R}$→$\mathbf{R}$ and $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$

(allows us to use L'Hôpital's Rule!)

Another basic list: (x = fixed number, k = konstant)

$\frac{1}{n} \to 0 \qquad k \to k \qquad x^{\frac{1}{n}} \to 1$

$n^{\frac{1}{n}} \to 1 \qquad (1 + \frac{x}{n})^n \to e^x \qquad \frac{x^n}{n!} \to 0$

$x^n \to \{$ 0, if $|x| < 1$; 1, if $x = 1$; diverges, otherwise $\}$

Infinite series

An infinite series is an infinite sum of numbers

$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$ (summation notation)

n-th term of series = a_n ; N-th partial sum of series = $s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** if the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n$; $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

if $|r| < 1$; otherwise, the series **diverges**.

Ex. Telescoping series: partial sums s_N 'collapse' to a compact expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

n-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the n-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ $\qquad$ $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty}(ka_n) = k \sum_{n=1}^{\infty} a_n$

Truncating a series: $\quad \sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

The integral test

Idea: $\quad \sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n, then the partial sums

$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ \mathrm{d}x \leq s_N = \sum_{n=a}^{N} a_n \leq \int_{a}^{N} f(x)\ \mathrm{d}x$$

so $\sum_{n=a}^{\infty} a_n$ converges exactly when $\int_{a}^{\infty} f(x)\ \mathrm{d}x$ converges

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ $\qquad$ (p-series)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges exactly when $p > 1$ $\qquad$ (logarithmic p-series?)

These families of series make good test cases for comparison with more involved terms (see below!)

Comparison tests

Again, think $\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$ $\quad$; $\quad$ if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\frac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1))$ diverges; L-comp with $\sum 1/(n \ln n)$

The ratio and root tests

Previous tests have you compare your series with **something else** (another series, an improper integral); these tests compare a series with itself (sort of)

Ratio Test: $\sum a_n$, $a_n \neq 0$ all n; $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Root Test: $\sum a_n$, $\lim\limits_{n\to\infty} |a_n|^{1/n} = L$

If $L < 1$ then $\sum a_n$ converges absolutely

If $L > 1$, then $\sum a_n$ diverges

If $L = 1$, then try something else!

Ex: $\sum \frac{4^n}{n!}$ converges by the ratio test $\qquad \sum \frac{n^5}{n^n}$ converges by the root test

Absolute convergence and alternating series

A series $\sum a_n$ <u>converges</u> <u>absolutely</u> if $\sum |a_n|$ converges.

If $\sum |a_n|$ converges then $\sum a_n$ converges. A series which converges but does not converge absolutely is called *conditionally convergent.*

An *alternating series* has the form $\sum(-1)^n a_n$ with $a_n \geq 0$ for all n.

If the sequence a_n is <u>decreasing</u> and has <u>limit</u> 0, then the **alternating series test** states that $\sum(-1)^n a_n$ converges. For example, $\sum_{n=0}^{\infty}(-1)^n/(n+1)$ converges, but not absolutely, so it is conditionally convergent.

The Basic Idea: For a series $\sum a_n$ to converge, it's n-th term a_n must go to 0. *But that isn't enough!* The n-th terms must go to 0 <u>fast</u> <u>enough</u> for their sum to remain finite. How fast is fast enough? That is exactly what the convergence tests are designed to determine....