Math 325, Section 1

Some practice problems for the final exam

1. (a) Use induction to show that for all $n \geq 1$, $n(n+1)$ is divisible by 2.

(b): Use induction to show that for every n≥1, $n^3 + 5n$ is divisible by 6.

2. Let $(a_n)_{n=1}^{\infty}$ be a <u>bounded</u> sequence of numbers, and define

$$s_n = \sup\{a_k : k \geq n\}$$

Show that the sequence $(s_n)_{n=1}^{\infty}$ is a monotonic sequence, and therefore converges. (Hint: write s_n as $\sup(A_n)$, and compare A_n to A_{n+1}.)

3. Prove, directly from the definition of a limit, that

$$\lim_{x \to 1}(x^2 - 3x + 1) = -1$$

5. Show that the function f:$\mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = x^3 + 3x - 7$$

has <u>exactly</u> one root between -1 and 2 (i.e., show it has at least one root, and <u>doesn't</u> have two!).

3. Given any two rational numbers $r_1 < r_2$, we can consider the linear function $f : [r_1, r_2] \to [0, 1]$ given by

$$f(x) = \frac{x - r_1}{r_2 - r_1}.$$

(a): Show that f is continuous, and if $x \in \mathbb{Q}$, then $f(x) \in \mathbb{Q}$, where $\mathbb{Q}$ = the rational numbers. (10 pts.)

(b): Show that between any two rational numbers $r_1 < r_2$ there is an **irrational** number. (Hint: $\sqrt{2}/2$ isn't rational (Why?); the intermediate value theorem could help!) (20 pts.)

4. Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be (bounded) sequences, and suppose that, for some subsequence $(b_{n_k})_{k=1}^{\infty}$, $a_k \leq b_{n_k}$ for all k. Show that

$$\lim_{n \to \infty} a_n \ \leq \ \lim_{n \to \infty} b_n$$

provided both limits exist! (Hint: You *can* take the high road, and just quote theorems, but there is also another way.) (20 pts.)

5. Show that if $(a_n)_{n=1}^{\infty}$ is a bounded sequence and $(a_{n_k})_{k=1}^{\infty}$ is a subsequence, then

$$\limsup_{k \to \infty}(a_{n_k}) \leq \limsup_{n \to \infty}(a_n)$$

(Hint: use problem 4! (although, again, there is another way.))

4. Suppose that $f : [0, 1] \to [0, \infty$ is a continuous function, with the property that, for all $x \in [0, 1]$, there is a $y \in [0, 1]$ such that $f(y) \leq (1/2)f(x)$. Show that there is a $c \in [0, 1]$

such that $f(c) = 0$. (Hint: use the hypothesis to find a sequence x_n in $[0, 1]$ such that $f(x_n) \to 0$.)

7. Let $f, g : D \to \mathbb{R}$ be two continuous functions. Show that the set

$$C = \{x \in D : f(x) = g(x)\}$$

contains any of its accumulation points that <u>also</u> lie in D.

8. Let $f : D \to \mathbb{R}$ be a function, and suppose that, for some $K \in \mathbb{R}$,

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in D$. Show that f is uniformly continuous.

9. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in $\mathbb{R}$. Define z_n by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Show that $(z_n)_{n=1}^{\infty}$ is also a Cauchy sequence <u>if</u> <u>and</u> <u>only</u> <u>if</u> the two original sequences converge to the same value.

11. Show that if $f, g : [a, b] \to \mathbb{R}$ are both integrable on [a,b], then so is fg.

(Hint: f and g are (by assumption) <u>both</u> bounded, and
$f(x)g(x) - f(y)g(y) = f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))$. This says something about $\sup\{f(x)g(x) : x \in [t_{i-1}, t_i]\} \ - \ \inf\{f(x)g(x) : x \in [t_{i-1}, t_i]\}$.)

8. Show that if $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, and $[c, d] \subseteq [a, b]$, then $f : [c, d] \to \mathbb{R}$ is also integrable on $[c, d]$.

(Hint: take a partition P of [a,b] with U(f,P)−L(f,P) small; then consider $Q = P \cup \{c, d\}$ (a partition of [a,b], and (more importantly) $S = Q \cap [c, d]$ (a partition of [c,d]). What can we say about $U(f, S) - L(f, S)$?) (Of course, there is also a different way!)

5. Show that if $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$, where $f_n, g_n : D \to \mathbb{R}$, are sequences of functions which converge **uniformly** to **bounded** functions $f, g : D \to \mathbb{R}$, then $f_n g_n$ converges uniformly to fg.
(Hint: **eventually**, the f_n and g_n are bounded; and

$$f_n g_n - fg = f_n(g_n - g) + g(f_n - f) \quad !)$$