Math 314/814, Section 5

Quiz number 9 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

By using the transpose A^T of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, find a basis for the column space of A, and use that basis to decide which of the systems of equations $A\vec{x} = \vec{b}$ are consistent, for $\vec{b}$ equal to

$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$.

Treating the column space of A as the row space of A^T, we can find a basis for $Col(A)$ by row reducing A^T:

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

This is in RREF, and so the (tranposes of the) non-$\vec{0}$ rows, $\begin{pmatrix} 1 \\ 0 \\ 3/2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix}$, are a basis for $\mathrm{Row}(A^T) = \mathrm{Col}(A)$.

But now our target vectors $\vec{b}$ give consistent systems precisely when they can be written

as a linear combination of our basis vectors, i.e. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 3/2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1/2 \end{pmatrix}$.

That is, $c = (3/2)a + (-1/2)b$. And we can test this!

$(3/2)(2) + (-1/2)(3) = 3/2 \neq 5$,so $A\vec{x} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ is inconsistent;

$(3/2)(5) + (-1/2)(2) = 13/2 \neq 3$, so $A\vec{x} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ is inconsistent;

$(3/2)(3) + (-1/2)(5) = 4/2 = 2$, so $A\vec{x} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ is consistent.