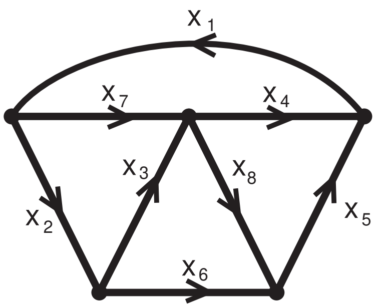
Math 314 Section 5

Quiz number 3 Solution

The graph below represents a network of pipes with water flowing (in the indicated directions) in them, with no leaks, and with flow rates indicated by the variables x_i. Find the system of linear equations for these flow variables, and find a (smallest) set of variables whose values determine the values of the remaining variables.

[Note: the order in which the variables were numbered was chosen with some care....]



The "flow in = flow out" equations for the five vertices are:

$$\begin{array}{lll}
x_1 = x_2 + x_7\ , & \text{which gives the linear equations} & x_1 - x_2 - x_7 = 0 \\
x_2 = x_3 + x_6 & & x_2 - x_3 - x_6 = 0 \\
x_3 + x_7 = x_4 + x_8 & & x_3 - x_4 + x_7 - x_8 = 0 \\
x_6 + x_8 = x_5 & & -x_5 + x_6 + x_8 = 0 \\
x_4 + x_5 = x_1 & & -x_1 + x_4 + x_5 = 0
\end{array}$$

This can be expressed as an augmented matrix (which we row reduce):

$$\left(\begin{array}{cccccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right) \longrightarrow \left(\begin{array}{cccccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0
\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & -1 & 0 & 0
\end{array}\right) \longrightarrow \left(\begin{array}{cccccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0
\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This is in row echelon form, with pivot variables x_1, x_2, x_3, x_5. Therefore, the free variables are x_4, x_6, x_7, x_8; these free variables determine the values of other, pivot, variables.