Solutions

Name:

Math 314 Exam 1

Show all work. Include all steps necessary to arrive at an answer unaided by a mechanical computational device. The steps you take to your answer are just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) For which value(s) of x does the system of linear equations, given by the augmented matrix

$$A = \left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 1 & 1 & 1 & 3 \\ 4 & 1 & x & 18 \end{array}\right),$$

have <u>**more than one**</u> solution?

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 1 & 1 & 1 & 3 \\ 4 & 1 & x & 18 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 0 & -2 & 2 & 4 \\ 4 & 1 & x & 18 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 0 & -2 & 2 & 4 \\ 0 & -11 & x+4 & 22 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -11 & x+4 & 22 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & x-7 & 0 \end{array}\right)$$

always consistent! will have a free variable precisely when $x-7=0$, i.e. $\boxed{x=7}$

So many solutions $\Longleftrightarrow \boxed{x=7}$.

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2. (20 pts.) In a model of an economy, there are three sectors, labeled E, P, and T, which purchase resources from one another:

sector E purchases 30% of the output of sector P, and 30% of the output of sector T;
sector P purchases 20% of the output of sector E, and 40% of the output of sector T;
sector T purchases 10% of the output of sector E, and 30% of the output of sector P.

The remaining output of each sector is purchased by that sector. If the total income for sector T is 100 (units), what must the incomes of the other sectors be in order for each sector to earn exactly what it spends?

costs: remainder

$E: \quad .3P + .3T + \boxed{}\,E \qquad E - .2E - .1E = .7E$

$P: \quad .2E + .4T + \boxed{}\,P \qquad P - .3P - .3P = .4P$

$T: \quad .1E + .3P + \boxed{}\,T \qquad T - .3T - .4T = .3T$

spends = earns:

$.7E + .3P + .3T = E$
$.2E + .4P + .4T = P$
$.1E + .3P + .3T = T$

$$\begin{array}{ccc} E & P & T \end{array}$$
$$\left(\begin{array}{ccc|c} -.3 & .3 & .3 & 0 \\ .2 & -.6 & .4 & 0 \\ .1 & .3 & -.7 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 2 & -6 & 4 & 0 \\ 1 & 3 & -7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -7 & 0 \\ 2 & -6 & 4 & 0 \\ -3 & 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -7 & 0 \\ 0 & -12 & 18 & 0 \\ 0 & 12 & -18 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -7 & 0 \\ 0 & -12 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -7 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -5/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

T is free $\quad E = \frac{5}{2}T$
$\qquad\qquad P = \frac{3}{2}T$

$T = 100 \implies E = \frac{5}{2}\cdot 100 = \boxed{250}$
$\qquad\qquad\quad P = \frac{3}{2}\cdot 100 = \boxed{150}$

(*) see note at end of exam (page 6).

3. (20 pts.) The vectors $\begin{bmatrix}1\\1\\1\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}$ do <u>not</u> span all of $\mathbb{R}^4$. Demonstrate this by showing that at least one of the standard coordinate vectors $\vec{e}_i$ in $\mathbb{R}^4$ does not lie in the span of these three vectors.

we decide if a vector is in the span by solving $(A|\vec{b})$

we'll do <u>all</u> $\vec{e}_i$ at once!

$$(A|I_4) = \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 & 0 \\ 0 & -3 & 2 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 & 0 \\ 0 & -3 & 2 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -4 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & -4 & 1 & -3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right)$$

The last row says that <u>none</u> of the vectors $\vec{e}_i$ give consistent systems, so <u>none</u> of them lie in the ~~sean~~ span of these three vectors.

4. (15 pts.) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} \text{ and } T\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, \text{ then what is } T\begin{bmatrix}1\\0\end{bmatrix}\ ?$$

Want to express $\begin{pmatrix}1\\0\end{pmatrix}$ as a lin combination of $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$!

$$\left(\begin{array}{cc|c}1&1&1\\2&-1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&1&1\\0&-3&-2\end{array}\right) \to \left(\begin{array}{cc|c}1&1&1\\0&1&2/3\end{array}\right) \to \left(\begin{array}{cc|c}1&0&1/3\\0&1&2/3\end{array}\right)$$

so $\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\-1\end{pmatrix}$

so $T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{3}T\begin{pmatrix}1\\2\end{pmatrix} + \frac{2}{3}T\begin{pmatrix}1\\-1\end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1/3+2/3\\-1/3+4/3\end{pmatrix} = \boxed{\begin{pmatrix}1\\1\end{pmatrix}}$

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or: $T(\vec{x}) = A\vec{x}$ for some matrix A. $T\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix} =$ first column of A.

so $A\begin{pmatrix}1&1\\2&-1\end{pmatrix} = \left(A\begin{pmatrix}1\\2\end{pmatrix}\ A\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}1&1\\-1&2\end{pmatrix}$

so $A = A\begin{pmatrix}1&1\\2&-1\end{pmatrix}\begin{pmatrix}1&1\\2&-1\end{pmatrix}^{-1} = \begin{pmatrix}1&1\\-1&2\end{pmatrix}\begin{pmatrix}1&1\\2&-1\end{pmatrix}^{-1} = \begin{pmatrix}1&1\\-1&2\end{pmatrix}\begin{pmatrix}1/3&1/3\\2/3&-1/3\end{pmatrix}$

$= \begin{pmatrix}1&0\\1&-1\end{pmatrix}$. so $\boxed{T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}}$

$$\left(\begin{array}{cc|cc}1&1&1&0\\2&-1&0&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&1&1&0\\0&-3&-2&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&1&1&0\\0&1&2/3&-1/3\end{array}\right) \to \left(\begin{array}{cc|cc}1&0&1/3&1/3\\0&1&2/3&-1/3\end{array}\right)$$

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5. ~~(20 pts.)~~ (25 pts.) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 & 6 \\ 3 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix},$$

and use this inverse to find solutions to the systems of equations $A\vec{x} = \vec{b}$, for

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 3 & 6 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 6 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & -4 & -9 & 0 & 1 & -3 \\ 0 & -1 & -2 & 1 & 0 & -2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & -4 & -9 & 0 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & -1 & -4 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 4 & -1 & -5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -16 & 4 & 21 \\ 0 & 1 & 0 & -9 & 2 & 12 \\ 0 & 0 & 1 & 4 & -1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & -9 & 2 & 12 \\ 0 & 0 & 1 & 4 & -1 & -5 \end{array}\right) \quad \text{So } \boxed{A^{-1} = \begin{pmatrix} 2 & 0 & -3 \\ -9 & 2 & 12 \\ 4 & -1 & -5 \end{pmatrix}}$$

Thus $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has solution $\vec{x} = \begin{pmatrix} 2 & 0 & -3 \\ -9 & 2 & 12 \\ 4 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}}$

$A\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ has ~~solutions~~ solution

$$\vec{x} = \begin{pmatrix} 2 & 0 & -3 \\ -9 & 2 & 12 \\ 4 & -1 & -5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4-9 \\ -18+2+36 \\ 8-1-15 \end{pmatrix} = \boxed{\begin{pmatrix} -5 \\ 20 \\ \cancel{12}\,-8 \end{pmatrix}}$$

$\boxed{\text{Check answers!}}$

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Check:

$$\begin{pmatrix} 2 & 3 & 6 \\ 3 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -2+15-12 \\ -3+10-6 \\ -1+10-8 \end{pmatrix} = \begin{pmatrix} 15-14 \\ 10-9 \\ 10-9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 2 & 3 & 6 \\ 3 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} -5 \\ 20 \\ -8 \end{pmatrix} = \begin{pmatrix} -10+60-48 \\ -15+40-24 \\ -5+40-32 \end{pmatrix} = \begin{pmatrix} 60-58 \\ 40-39 \\ 40-37 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \checkmark$$

Addendum to #2!

Several of you took the tack that $T=100$ should appear in the equations. This led to either

$$\boxed{\begin{aligned} -.3E + .3P &= -30 \\ .2E - .6P &= -40 \\ .1E + .3P &= 70 \end{aligned}}$$

or

$$\boxed{\begin{aligned} -.3E + .3P + .3T &= 0 \\ .2E - .6P + .4T &= 0 \\ .1E + .3P + .3T &= 100 \end{aligned}}$$ (← a partial use of $T=100$.)

As it happens, both of these work! (I was rather surprised that the partial use of $T=100$ did!)