Math 314/814 Matrix Theory

Exam 1 Solutions

1. (20 pts.) Show whether or not the system of linear equations given by the augmented matrix

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 2 & 4 & 1 & | & 2 \\ 3 & 1 & 7 & -6 & | & -3 \\ 2 & 5 & 9 & 4 & | & 7 \end{pmatrix}$$

has a solution. If it does, does it have one or more than one solution?

We row reduce! As always, there is more than one correct way to do so.

$$\begin{pmatrix} 1 & 2 & 4 & 1 & | & 2 \\ 3 & 1 & 7 & -6 & | & -3 \\ 2 & 5 & 9 & 4 & | & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 1 & | & 2 \\ 0 & -5 & -5 & -9 & | & -9 \\ 0 & 1 & 1 & 2 & | & 3 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 2 & 4 & 1 & | & 2 \\ 0 & 1 & 1 & 2 & | & 3 \\ 0 & -5 & -5 & -9 & | & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & -3 & | & -4 \\ 0 & 1 & 1 & 2 & | & 3 \\ 0 & 0 & 0 & 1 & | & 6 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & 2 & 0 & | & 14 \\ 0 & 1 & 1 & 0 & | & -9 \\ 0 & 0 & 0 & 1 & | & 6 \end{pmatrix}$$

Since the coefficient matrix has no row of zeros, the system is consistent, so there is a solution. Since the thrid column contains no pivot, the system has a free variable, so there is more than one solution.

More than asked: setting the free variable equal to zero yields the solution $\begin{pmatrix} 14 \\ -9 \\ 0 \\ 6 \end{pmatrix}$.

2. (20 pts.) The figure below models a network of pipes, with numbers indicating the flow rate (in the direection of each arrow) on some pipes, and variables indicating unknown flow rates. Are the known rates sufficient to determine all of the remaining rates? If so, determine the unknown rates; if not, what further values would be sufficient to determine the remaining rates?

[Figure omitted.]

At each vertex we set up an equation (in) = (out), and solve:

$7 = 5 + a$, $a + b = 6$, $5 = c + 2$, $2 + d = b$, $c + 6 = d + 7$, or

$a = 2$, $a + b = 6$, $c = 3$, $b - d = -2$, $c - d = 1$

These can be solved by substitution; here we'll row reduce the matrix!

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 1 & 1 & 0 & 0 & | & 6 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 1 & -1 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 1 & -1 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & -1 & | & -2 \\ 0 & 0 & 1 & -1 & | & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & -1 & | & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The row of zeroes at the bottom tells us that the system is consistent. There is a pivot in every column, so there are no free variables, so there is only one solution. So the known rates do determine all of the unknown rates. These can be read off from the RREF above:

$$a = 2 \quad , \quad b = 4 \quad , \quad c = 3 \quad , \quad d = 2 \; .$$

3. (25 pts.) Is the vector $\vec{b} = \begin{bmatrix} 7 \\ 21 \\ 1 \end{bmatrix}$ in the span of the vectors

$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$? Are the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent?

Translating, we wish to know if the system below is consistent, which we determine by row reducing:

$$\begin{pmatrix} 2 & 1 & 4 & | & 7 \\ 3 & 5 & -1 & | & 21 \\ -1 & 1 & -5 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 5 & | & -1 \\ 2 & 1 & 4 & | & 7 \\ 3 & 5 & -1 & | & 21 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 5 & | & -1 \\ 0 & 3 & -6 & | & 9 \\ 0 & 8 & -16 & | & 24 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 5 & | & -1 \\ 0 & 1 & -2 & | & 3 \\ 0 & 8 & -16 & | & 24 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 5 & | & -1 \\ 0 & 1 & -2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & | & 2 \\ 0 & 1 & -2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The row of zeroes at the bottom tells us that the system is consistent, so $\vec{b}$ is in the span of $\vec{v}_1$,$\vec{v}_2$,$\vec{v}_3$. To decide if these vectors are linearly independent, we would row reduce the same coefficient matrix, except with the 0-vector as target, yielding

$$\begin{pmatrix} 2 & 1 & 4 & | & 0 \\ 3 & 5 & -1 & | & 0 \\ -1 & 1 & -5 & | & 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & 3 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Since there is no pivot in the third column, this system of equations has more than one solution, so the three vectors are not linearly independent.

4. (20 pts.) Use Gauss-Jordan elimination to find the inverse of the matrix A, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 7 & 8 \\ 2 & 7 & 4 \end{pmatrix}$$

Row reducing the superaugmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 7 & 8 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 3 & -2 & -2 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 7 & -3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 9 & -3 \\ 0 & 1 & 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & 7 & -3 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -28 & 13 & -5 \\ 0 & 1 & 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & 7 & -3 & 1 \end{array}\right), \quad \text{so} \quad A^{-1} = \begin{pmatrix} -28 & 13 & -5 \\ 4 & -2 & 1 \\ 7 & -3 & 1 \end{pmatrix}$$

(b) (5 pts.) Use your answer from part (a) to find the solution to the equation $Ax = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$Ax = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ has solution

$$x = A^{-1}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -28 & 13 & -5 \\ 4 & -2 & 1 \\ 7 & -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -84+26-5 \\ 12-4+1 \\ 21-6+1 \end{pmatrix} = \begin{pmatrix} -63 \\ 9 \\ 16 \end{pmatrix}$$

5. (10 pts.) Find an example of matrices A and B so that $A\vec{u} = \vec{0}$ and $B\vec{v} = \vec{0}$ both have non-trivial solutions, but $(A+B)\vec{w} = \vec{0}$ has no non-trivial solution.

This question has an immense number of solutions. For example,

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$, which is in RREF and has a pivot in every column, so $I_2\vec{w} = \vec{0}$ has no non-trivial solutions.

Another:

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} = A$, which row reduces to

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has no free variables, so $A\vec{w} = \vec{0}$ has no non-trivial solutions.

Another!

Take your favorite invertible matrix A, and replace its last column with a column of zeroes (yielding B), and build another matrix (C) from the all-0 matrix by replacing its last column (of 0s) with the last column of your invertible matrix A. Then $B + C = A$, and $A\vec{w} = \vec{0}$ has only the solution $\vec{w} = A^{-1}\vec{0} = \vec{0}$, but $B[0, \ldots, 0, 1]^T = \vec{0}$ and $C[1, 0, \ldots, 0]^T = \vec{0}$.