Math 314/814, Section 1

Quiz number 8 Solutions

Find the determinant of the matrix

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & -2 & 3 \end{pmatrix}$$

(a) by expanding along a row or column of your choice,
or
(b) by row reducing the matrix A.

Expanding on the third row, we have

$$\det(A) = (-1)^{3+1}2\begin{vmatrix} 1 & 4 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 3 \end{vmatrix} + 0 + 0 + (-1)^{3+4}3\begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2\{(-1)^{2+1}1\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} + 0 + (-1)^{2+3}2\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix}\}$$

$$- 3\{(-1)^{2+1}2\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} + (-1)^{2+2}1\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} + 0\}$$

$$= 2\{(-1)(12+2) - (2)(-2-4)\} - 3\{(-2)(-2-4) + (1)(-6-4)\}$$

$$= 2(-14+12) - 3(12-10) = 2(-2) - 3(2) = -4 - 6 = -10\ .$$

By row reduction, keeping track of the steps to assess their effect on the determinant:

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & -2 & 3 \end{pmatrix} \rightarrow_{E_{4,1}} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 3 & 1 & 4 & 1 \end{pmatrix} \rightarrow_{E_{2,1}(-2),E_{3,1}(-2),E_{4,1}(-3)}$$

$$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -4 \\ 0 & -2 & 4 & -3 \\ 0 & -2 & 10 & -8 \end{pmatrix} \rightarrow_{E_2(-1)} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & -2 & 4 & -3 \\ 0 & -2 & 10 & -8 \end{pmatrix} \rightarrow_{E_{3,2}(2),E_{4,2}(2)}$$

$$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & 2 & 0 \end{pmatrix} \rightarrow_{E_3(1/2),E_{3,4}} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 5 \end{pmatrix} \rightarrow_{E_{4,3}(4)} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\rightarrow_{E_4(1/5)} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow_{E_{i,j}(r)'\text{s}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so
$1 = \det(I_4) = (-1)(1)(1)(1)(-1)(1)(1)(1/2)(-1)(1)(1/5)(1)\det(A) = (-1/10)\det(A)$,
so $\det(A) = -10$.

Math 314/814, Section 5

Quiz number 8 Solutions

Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 3 & 1 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

(a) by expanding along a row or column of your choice,
or
(b) by row reducing the matrix A.

Expanding on the third row, we have

$$\det(A) = (-1)^{3+1}3\begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} + 0 + 0 + (-1)^{3+4}1\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 3\{(-1)^{2+1}1\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} + 0 + (-1)^{2+3}2\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}\}$$

$$- 1\{(-1)^{2+1}3\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} + (-1)^{2+2}1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} + 0\}$$

$$= 3\{(-1)(4-2) - (2)(4-4)\} - \{(-3)(4-4) + (1)(2-4)\}$$

$$= (3)(-2-0) - (0+(-2)) = -6+2 = -4\ .$$

By row reduction, keeping track of the steps to assess their effect on the determinant:

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 3 & 1 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \rightarrow_{E_{2,1}(-3),E_{3,1}(-3),E_{4,1}(-1)} \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & -5 & -12 & -1 \\ 0 & -6 & -12 & -2 \\ 0 & -1 & -2 & 0 \end{pmatrix} \rightarrow_{E_{4,2},E_2(-1)}$$

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & -2 \\ 0 & -5 & -12 & -1 \end{pmatrix} \rightarrow_{E_{3,2}(6),E_{4,2}(5)} \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \rightarrow_{E_{4,3},E_3(-1/2),E_4(-1/2)}$$

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow_{E_{i,j}(r)'\text{s}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so
$1 = \det(I_4) = (1)(1)(1)(-1)(-1)(1)(1)(-1)(-1/2)(-1/2)(1)\det(A) = (-1/4)\det(A)$,
so $\det(A) = -4$.