Math 314/814 Matrix Theory

Solutions to some of the Exam 2 practice problems

2. Find a basis for $\mathbb{R}^3$ from among the vectors

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \text{ which \underline{includes} the vector } \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

In order to ensure that $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is included, we can write it as the first column of a matrix

$A = \begin{bmatrix}1&1&3&-1\\1&2&2&2\\1&3&1&1\end{bmatrix}$, and row reduce:

$$\begin{bmatrix}1&1&3&-1\\1&2&2&2\\1&3&1&1\end{bmatrix} \to \begin{bmatrix}1&1&3&-1\\0&1&-1&3\\0&2&-2&2\end{bmatrix} \to \begin{bmatrix}1&1&3&-1\\0&1&-1&3\\0&0&0&-4\end{bmatrix} \to \begin{bmatrix}1&1&3&-1\\0&1&-1&3\\0&0&0&1\end{bmatrix}$$

so we have pivots in the 1st, 2nd, and 4th columns, and so the corresponding columns of A,

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix},$$

form a basis for col(A). But since there are 3 of them, they also form a basis for $\mathbb{R}^3$.

3. Suppose T:$\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, and suppose that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}, T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\-2\\1\end{bmatrix}, \text{ and } T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-3\\5\end{bmatrix}.$$

What is $T\begin{bmatrix}2\\3\\-5\end{bmatrix}$?

$$\begin{bmatrix}2\\3\\-5\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + (-5)\begin{bmatrix}0\\0\\1\end{bmatrix}, \text{ so , since } T \text{ is linear,}$$

$$T\begin{bmatrix}2\\3\\-5\end{bmatrix} = T\Big(2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + (-5)\begin{bmatrix}0\\0\\1\end{bmatrix}\Big) = 2T\begin{bmatrix}1\\0\\0\end{bmatrix} + 3T\begin{bmatrix}0\\1\\0\end{bmatrix} + (-5)T\begin{bmatrix}0\\0\\1\end{bmatrix} =$$

$$2\begin{bmatrix}-1\\2\\3\end{bmatrix} + 3\begin{bmatrix}3\\-2\\1\end{bmatrix} + (-5)\begin{bmatrix}1\\-3\\5\end{bmatrix} = \begin{bmatrix}-2+9-5\\4-6+15\\6+3-25\end{bmatrix} = \begin{bmatrix}2\\13\\-16\end{bmatrix}$$

4. Find the determinant of the matrix $B = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 5 & 2 & 2 \\ 0 & 2 & 3 & 4 \\ 3 & 3 & 1 & 3 \end{bmatrix}$.

By row reduction:

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 5 & 2 & 2 \\ 0 & 2 & 3 & 4 \\ 3 & 3 & 1 & 3 \end{bmatrix} \overset{(1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & 3 & 4 \\ 0 & -6 & -5 & -3 \end{bmatrix} \overset{(-1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & -6 & -5 & -3 \end{bmatrix} \overset{(1)}{\rightarrow}$$

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 7 & 9 \end{bmatrix} \overset{(-1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 9 \end{bmatrix} \overset{(1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

ending in a diagonal matrix, so

$\det(B) = (1)(-1)(1)(-1)(1)[(1)(1)(1)(9)] = 9.$

Or, by expanding along the 1st column:

$$\det(B) = (1)\begin{vmatrix} 5 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 3 \end{vmatrix} - (2)\begin{vmatrix} 3 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 3 \end{vmatrix} + (0)\begin{vmatrix} 3 & 2 & 2 \\ 5 & 2 & 2 \\ 3 & 1 & 3 \end{vmatrix} - (3)\begin{vmatrix} 3 & 2 & 2 \\ 5 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= (1)((5)\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + (3)\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}) - (2)((3)\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + (3)\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}) -$$
$$(3)((3)\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} - (5)\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + (2)\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix})$$

$$\begin{aligned}
&= (1)[(5)(5)-(2)(4)+(3)(2)]-(2)[(3)(5)-(2)(4)+(3)(2)]-(3)[(3)(2)-(5)(2)+(2)(0)] \\
&= (1)[25-8+6]-(2)[15-8+6]-(3)[6-10+0] \\
&= (1)[23]-(2)[13]-(3)[-4] \\
&= 23-26+12 \\
&= 9
\end{aligned}$$

(Other sequences of row reductions, and other expansions, are also possible.)

5. The matrix $A = \begin{bmatrix} 3 & -8 & 3 \\ 2 & -14 & 6 \\ 2 & -31 & 14 \end{bmatrix}$ has characteristic polynomial
$\chi_A(\lambda) = -(\lambda+1)(\lambda-2)^2$.

Find the eigenvalues of A and bases for each of the corresponding eigenspaces.
Is A diagonalizable? Why or why not?

Eigenvalues are solutions to $\chi_A(\lambda) = \det(A - \lambda I) = -(\lambda+1)(\lambda-2)^2 = 0$,
so $\lambda = -1$ or $\lambda = 2$.

E_{-1}:

$$A-(-1)I = \begin{bmatrix} 4 & -8 & 3 \\ 2 & -13 & 6 \\ 2 & -31 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3/4 \\ 2 & -13 & 6 \\ 2 & -31 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3/4 \\ 0 & -9 & 9/2 \\ 0 & -27 & 27/2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 3/4 \\ 0 & 1 & -1/2 \\ 0 & -27 & 27/2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x - z/4 = 0$, $y - z/2 = 0$, so $4x = z = 2y$, so, e.g., $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is a basis for E_{-1}.

E_2:

$$A-2I = \begin{bmatrix} 1 & -8 & 3 \\ 2 & -16 & 6 \\ 2 & -31 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -8 & 3 \\ 0 & 0 & 0 \\ 0 & -15 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -8 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -2/5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1/5 \\ 0 & 0 & 0 \\ 0 & 1 & -2/5 \end{bmatrix}$$

so $x - z/5 = 0$, $y - 2z/5 = 0$, so $5x = z$, $5y = 2z$, so e.g., $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ is a basis for E_2.

Since $\mathbb{R}^3$ does <u>not</u> have a basis consisting of eigenvectors for A
(or: the geometric multiplicity of $\lambda = 2$ is $1 < 2 =$ the algebraic multiplicity of $\lambda = 2$),
A is not diagonalizable.