Math 314/814, Section 6

Quiz number 9 Solutions

Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, and find bases for each of the associated eigenspaces.

To find the eigenvalues, we compute the characteristic polynomial:

$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - (4)(3) = \lambda^2 - 3\lambda + 2 - 12$
$= \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$

So the eigenvalues are the solutions to $\chi_A(\lambda) = (\lambda - 5)(\lambda + 2) = 0$, i.e., $\lambda = 5$ and $\lambda = -2$. From this we can find the associated nullspaces and bases, by row reduction:

$$A - 5I = \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4/3 \\ 3 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4/3 \\ 0 & 0 \end{pmatrix}.$$

The second column represents the free variable, so vectors in the nullspace satisfy $x - (4/3)y = 0$, so $x = (4/3)y$, and so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (4/3)y \\ y \end{pmatrix} = y \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$, and so $\begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$ is a basis for the 5-eigenspace of A.

In the same way, $A - (-2)I = A + 2I = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Again, the second column represents the free variable, so vectors in the nullspace satisfy $x + y = 0$, so $x = -y$. and so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and so $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a basis for the (-2)-eigenspace of A.

Note: There <u>are</u> many other bases for these eigenspaces, like $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. or $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ for the 5-eignespace; generally, we can multiply basis vectors by (non-zero) scalars and still have basis vectors...

Note also that we can check our answers!

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4/3 + 4 \cdot 1 \\ 3 \cdot 4/3 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 8/3 + 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 20/3 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-1) + 4 \cdot 1 \\ 3 \cdot (-1) + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = (-2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$