Math 314/814, Section 6

Quiz number 7 AS IT SHOULD HAVE BEEN Solutions

Use a superaugmented matrix to find a/the matrix B whose nullspace is equal to the column space of the matrix

$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 7 \\ 2 & -1 & 8 \end{pmatrix}$, and use this to decide if the linear system $A\vec{x} = \vec{b}$

is consistent, for each of the vectors $\vec{b} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

To find the matrix B, we row reduce!

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 1 & -2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 8 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 7 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 2 & -1 & 8 & 0 & 0 & 1 \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 7 & 0 & 1 & 0 \\ 0 & 5 & -10 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 & -2 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 7 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1/5 & -2/5 & 0 \\ 0 & 3 & -6 & 0 & -2 & 1 \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 7 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1/5 & -2/5 & 0 \\ 0 & 0 & 0 & -3/5 & -4/5 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 7 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1/5 & -2/5 & 0 \\ 0 & 0 & 0 & -3 & -4 & 5 \end{array}\right)$$

[The last step was for mostly cosmetic reasons.]

This means that consistent systems $A\vec{x} = \vec{b}$ require

$(\,-3 \quad -4 \quad 5\,)\,\vec{b} = (0)$ [in order for the REF to have last row $(\,-0 \quad 0 \quad 0 \quad | \quad 0\,)$] ,

that is, $\mathrm{Col}(A) = \mathrm{Null}(\,-3 \quad -4 \quad 5\,)$. Then since

$(\,-3 \quad -4 \quad 5\,)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = -9 - 16 + 25 = 0,\ A\vec{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ is consistent!

$(\,-3 \quad -4 \quad 5\,)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -6 - 12 + 20 = 2 \neq 0,\ A\vec{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is not consistent.

$(\,-3 \quad -4 \quad 5\,)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -3 - 8 + 15 = 4 \neq 0,\ A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not consistent.