Math 310  Homework #1  Solutions

1. Show \( \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 = \frac{1}{4} n^2(n^2+2n+1) = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2 \)

By induction:

(i) \( n = 1 \) \( \sum_{k=1}^{1} k^3 = 1 \) \( \left( \frac{1(1+1)}{2} \right)^2 = \left( \frac{2}{2} \right)^2 = 1 \) \( \checkmark \)

(ii) Suppose \( \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 \). Then

\[
\sum_{k=1}^{n+1} k^3 = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n^3+3n^2+3n+1) + \left( \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2 \right)
\]

But \( \left( \frac{n(n+1)}{2} \right)^2 = \left( \frac{n^2+n+2}{2} \right)^2 = \frac{1}{4} (n^2+3n+2)^2 = \frac{1}{4} (n^4+3n^3+3n^2+2n+4) = \frac{1}{4} (n^4+6n^3+13n^2+12n+4) = \frac{1}{4} n^4 + \frac{3}{4} n^3 + \frac{3}{4} n^2 + \frac{3}{4} n + 1 \)

So \( \sum_{k=1}^{n+1} k^3 = \left( \frac{n(n+1)(n+2)}{2} \right)^2 \) \( \checkmark \)

So by P.M.I., \( \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 \) for all \( n \geq 1 \).

2. Show \( 4.5^n + 7.27^n \) is a multiple of 11 for all \( n \geq 0 \)

By induction:

(i) \( n = 0 \) \( 4.5^0 + 7.27^0 = 4 + 7 = 11 = 11 \cdot 1 \) \( \checkmark \)

(ii) Suppose \( 4.5^n + 7.27^n = 11k \) for some integer \( k \).

Then \( 4.5^{n+1} + 7.27^{n+1} = (4.5^n) \cdot 5 + (7.27^n) \cdot 27 \)

\begin{align*}
= 5 \cdot (4.5^n + 7.27^n) + (27-5) \cdot (7.27^n) \\
= 5 \cdot (11k) + 22 \cdot (7.27^n) = 11 \cdot (5k + 2.7.27^n)
\end{align*}

is a multiple of 11.
So, by P.M.I., \(4 \cdot 5^n + 7 \cdot 27^n\) is a multiple of \(n\) for all \(n \geq 0\).

3. Show \(55 \cdot 44^n - 6 \cdot 23^n\) is a multiple of \(7\) for all \(n \geq 0\).

By induction:

(i) \(n = 0\)
\[
55 \cdot 44^0 - 6 \cdot 23^0 = 55 - 6 = 49 = 7 \cdot 7 \quad \checkmark
\]

(ii) If \(55 \cdot 44^n - 6 \cdot 23^n = 7k\), then
\[
55 \cdot 44^{n+1} - 6 \cdot 23^{n+1} = 44 \cdot (55 \cdot 44^n) - 23 \cdot (6 \cdot 23^n)
\]
\[
= 23 \left(55 \cdot 44^n - 6 \cdot 23^n\right) + (44 - 23) \cdot (6 \cdot 23^n)
\]
\[
= 23 \cdot (7k) + (21) \cdot (6 \cdot 23^n) = 7 \cdot (23k + 3 \cdot 6 \cdot 23^n)
\]

is also a multiple of \(7\). \(\checkmark\)

So by P.M.I., \(55 \cdot 44^n - 6 \cdot 23^n\) is a multiple of \(7\), for all \(n \geq 0\).

4. For every odd \(m \geq 1\), \(4^m + 5^m\) is a multiple of \(9\).

\(m\) is odd means \(m = 2k + 1\), \(m \geq 1\) means \(k \geq 0\). So we want: For all \(k \geq 0\) \(4^{2k+1} + 5^{2k+1}\) is a multiple of \(9\).

Prove by induction!

(i) \(k = 0\)
\[
4^{2\cdot 0+1} + 5^{2\cdot 0+1} = 4^1 + 5^1 = 4 + 5 = 9 = 9 \cdot 1 \quad \checkmark
\]

(ii) If \(4^{2k+1} + 5^{2k+1} = 9 \cdot l\), then
\[
4^{2(k+1)+1} + 5^{2(k+1)+1} = 4^{(2k+1)+2} + 5^{(2k+1)+2}
\]
\[
= 16 \left(4^{2k+1}\right) + 25 \left(5^{2k+1}\right) = 16 \left(4^{2k+1} + 5^{2k+1}\right) + (25-16)(5^{2k+1})
\]
\[
= 16 \left(9l\right) + 9 \left(5^{2k+1}\right) = 9 \left(16l + 5^{2k+1}\right)
\]

is a multiple of \(9\). \(\checkmark\)

So, by P.M.I., \(4^m + 5^m\) is a multiple of \(9\) for all odd \(m \geq 1\).
5. For any convex polygon with \( n \) sides, the sum of the interior angles is \((n-2)\pi\).

By complete induction:

(i) smallest \( n \) making a polygon is \( n=3 \) (triangle). The sum of the angles of a triangle is \( \pi = (3-2)\pi \) (from high school geometry).

(ii) Suppose that every polygon with \( 3 \leq k < n \) sides has sum of interior angles = \( (k-2)\pi \). Then for a convex polygon with \( n \) sides, \( n \geq 3 \), with 3 adjacent vertices: \( A, B, C \), draw the line segment \( AC \). This cuts the convex polygon (call it \( P \)) into two convex polygons \( P' \) and \( P'' \), one having \( (n-1) \) sides (\( P' \), say), and one having \( 3 \leq n \) sides (\( P'' \), say). By our inductive hypothesis, the sum of the interior angles of \( P' \) is \( ((n-1)-2)\pi = (n-3)\pi \), and the sum of the interior angles of \( P'' \) is \( \pi \). But! from the picture, the interior angles of \( P \) and \( P'' \) together add up to the interior angles of \( P' \) (since the sum of the interior angles of \( P \) is \( (n-3)\pi + \pi = (n-2)\pi \)).

So by complete induction, the sum of the angles of a \( n \) polygon with \( n \) sides is \((n-2)\pi\).

[FYI: This result is also true for polygons that aren't convex, but you need to be much more careful; you need to allow "reflex" angles \( (>\pi) \) inside \( P \) and you to worry that the line segment \( AC \) hits \( P \).]

and you need to worry that \( AC \) is outside of \( P \):
6. $S \subseteq \mathbb{Z}$ so that for some $N \in \mathbb{Z}$, $s \leq N$ for all $s \in S$. (Then $S$ has a largest element, i.e. $\exists \bar{s} \in S$ so that $s \leq \bar{s}$ for all $s \in S$.

Note: need $S \neq \emptyset$, otherwise "$\exists \bar{s} \in S$" (forget the rest...) is impossible.

If $S \neq \emptyset$ and $N$ is as above, set $A = \{N-s, \text{where } s \in S\}$ then $A \neq \emptyset$ (there's at least one $N-s$), and since $s \leq N$ for all $s \in S$, $N-s \geq 0$ for all $s \in S$, so $a \geq 0$ for all $a \in A$, i.e. $A \subseteq \mathbb{N}$. Then by well-ordering, $A$ has a smallest element $a_0$, i.e. $a_0 \in A$ and $a_0 \leq a$ for all $a \in A$. But then $a_0 = N - s_0$ for some $s_0 \in S$. Then for any $s \in S$, $N - s = a \in A$, so $N - s = a_0 = a = N - s$, so $N - s \leq N - s_0$, so $N - s + (s + s_0) = N + s \leq N - s + (s + s_0) = N + s_0$, so $N + s - N = s \leq N + s - N = s_0$, i.e. $s \leq s_0$ for all $s \in S$.

So $s_0$ is the largest element of $S$. \[\]