Math 221

Topics for the third exam

Chapter 3: Second Order Linear Equations

§8: **Spring - Mass problems**

Basic setup: an object with mass m suspended on a spring. At rest, the mass stretches the spring by a length L. The mass is then displaced from this equilibrium position and released (with some initial velocity). Position at time t is $u(t)$.

Newton: $mu'' =$ sum of the forces acting on the object. These include:

gravity: $F_g = mg$

the spring: $F_s = -k(u+L)$ (Hooke's Law)

friction: $F_f = -\gamma u'$

a possible external force: $F_e = f(t)$

Equilibrium: gravity and spring forces balance out; $mg - kL = 0$ (use to compute k !)

So: $mu'' = -ku - \gamma u' + f(t)$, i.e.,

$$mu'' + ku + \gamma u' = f(t)$$

Some special cases:

No friction ($\gamma = 0$) = undamped, no external force (= free vibration); solutions are

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C \cos(\omega_0 t - \delta)$$

where $\omega = \sqrt{k/m}$ = the *natural frequency* of the system, C = amplitude of the vibration, δ (= 'delay') = phase angle

$C = \sqrt{(c_1^2 + c_2^2)}$, $\tan(\delta) = c_2/c_1$

$T = 2\pi/\omega_0$ = period of the vibration. Note: stiffer spring (= larger k) gives higher frequency, shorter period. Larger m gives the opposite.

Damped free vibrations; solutions depend on $\gamma^2 - 4km$ = discriminant

$\gamma^2 > 4km$ (overdamped); fundamental solutions are $e^{r_1 t}, e^{r_2 t}$, $r_1, r_2 < 0$

$\gamma^2 = 4km$ (critically damped); fundamental solutions are e^{rt}, te^{rt}, $r < 0$

$\gamma^2 < 4km$ (underdamped); fundamental solutions are $e^{rt}\cos(\omega t), e^{rt}\sin(\omega t)$, $r < 0$, $\omega = \sqrt{\omega_0^2 - (\gamma/2m)^2}$

In each case, solutions tend to 0 as t goes to ∞. In first two cases, the solution has at most on local max or min; in the third case, note that the frequency of the periodic part of the motion is smaller than the natural frequency. $T = 2\pi/\omega$ is called the *quasi-period* of the vibration.

§9: **Forced vibrations**

Focus on periodic forcing term: $f(t) = F_0 \cos(\omega t)$.

Undamped: if $\omega \neq \omega_0$, then (using undetermined coefficients) solution is

$$u = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

This is the sum of two vibrations with different frequencies.

In the special case $u(0) = 0, u'(0) = 0$ (starting at rest), we can further simplify:

$$u = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin(\frac{\omega_0 - \omega}{2}t)\sin(\frac{\omega_0 + \omega}{2}t)$$

When ω is close to ω_0, this illustrates the concept of *beats*; we have a high frequency vibration (the second sine) with *amplitude* a low frequency vibration (the first sine). the mass essentially vibrates rapidly between to sine curves.

When $\omega = \omega_0$, our forcing term is a solution to the homogeneous equation, so the general solution, instead, is

$$u = C\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t\sin(\omega t)$$

In this case, as t goes to ∞, the amplitude of the second oscillation goes to ∞; the solution, essentially, resonates with the forcing term. (Basically, you are 'feeding' the system at it's natural frequency.) This illustrates the phenomenon of *resonance*.

Finally, if we include friction ($\gamma \neq 0$), then the solution turns out to be

$$u = \text{homog. soln.} + C\cos(\omega t - \delta), \text{ where}$$

$$C = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},\ \tan(\delta) = \frac{\gamma m}{m(\omega_0^2 - \omega^2)}$$

But since $\gamma > 0$, the homogeneous solutions will tend to 0 as $t \to \infty$; they are called the *transient solution*. (Basically, they just allow us to solve any initial value problem. We can then conclude that any energy given to the susystem is dissipated over time; leaving only the energy imparted by the forcing term to drive the system along.) The other term is called the *forced response*, or *steady-state solution*.

Note that the amplitude C of the forced response goes to 0 as the driving frequency, ω, goes to ∞. Notice also that $\tan(\delta)$ can never be 0, so the forced response is always out of phase with the forcing term. When we are driving the system at it's natural frequency ω_0, the system is 90 degrees out of phase; as $\omega \to \infty$, the system approaches being 180 degrees out of phase, i.e., the motion of the mass is almost exactly <u>opposite</u> to the force being externally applied!

Chapter 7: Systems of first order linear equations

§1: **Introduction**

Basic idea: we have *two* (or more) quantities, with their rates of change depending upon *one another*.

Ex: multiple spring - mass system:

WALL - spring - mass - spring - mass - spring - WALL   walls are A units apart

u_1 = position of first mass, u_2 = position of second mass, then we find , by analyzing the forces involved, that

$m_1 u_1'\prime = -k_1 u_1 + k_2(u_2 - u_1) - \gamma_1 u_1'$ hskip.2in $m_2 u_2'\prime = -k_2(u_2 - u_1) + k_3(A - u_2) - \gamma_2 u_2'$

where the symbols have similar meanings to our ordinary situation. The appropriate notion of an initial value problem would be to know the initial positions and initial velocities of both masses at a fixed time.

Ex: mixing with multiple tanks:

Tank 1 flows to tank 2 with rate r_1, 2 to 3 with rate r_2, and 3 to 1 with rate r_3, then if the u's are the amount of solute in each tank, and the V's are the volumes in each, we have

$$u_1' = \frac{r_3}{V_3}u_3 - \frac{r_1}{V_1}u_1, u_2' = \frac{r_1}{V_1}u_1 - \frac{r_2}{V_2}u_2, u_3' = \frac{r_2}{V_2}u_2 - \frac{r_3}{V_3}u_3$$

Here the appropriate notion of *initial value problem* is to know the values of each of u_1, u_2, u_3 at a fixed time.

An important class of examples: any ordinary differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

can be turned into a system of *first order* equations by setting

$$u_1 = y, u_2 = y', \dots, u_{n-1} = y^{(n-1)}$$

giving the system of equations

$$u_1' = u_2, u_2' = u_3, \dots, u_{n-1}' = f(t, y, y', \dots, y^{(n-1)})$$

§2: **Matrices**

The best way to develop techniques for solving systems of equations is with *matrices.* Notationally, a system of equations

$$u_1' = 2u_1 + 3u_2 \qquad u_2' = 3u_1 - u_2$$

can be expressed as

$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, or, writing $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and using the derivative of a vector-valued function, $\mathbf{u}' = \begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix} \mathbf{u} = A\mathbf{u}$. The square of numbers is called a (coefficient) *matrix.* We actually want to think of $A\mathbf{u}$ as a real multiplication; the basic idea is that the i-th entry of $A\mathbf{u}$ is the i-th row of A *times* $\mathbf{u}$. This in turn should be familiar as the dot product of the row of A and $\mathbf{u}$.

More generally, we can multiply matrices, getting another matrix. Using the notation $A = (a_{ij})$, where a_{ij} is the entry in the i-th row and j-th column of A, we write

$AB = (c_{ij})$, where $c_{ij} =$ the dot product of the i-th row of A and the j-th column of B .

Amazingly, this product has alot of the properties we are used to: if we add matrices by adding their ij-th entries (to get the ij-th entry of the sum), and multiply by a scalar c by multliplying each entry by c, then we have:

$$(AB)C = A(BC) \text{ , } A(B + C) = AB + AC \text{ , } A(cB) = c(AB) \text{ , } c(A + B) = cA + cB$$

However what we do not NOT **NOT** have is $AB = BA$; usually, matrix multiplication does not NOT **NOT** commute! We will need the special matrix $I = (\delta_{ij})$, where $\delta_{ij} = 1$ if $i = j$, and 0 if $i \neq j$. This matrix has the property that $IA = AI = A$ for all matrices A, i.e., it acts like the number 1 under multiplication.

If A is a matrix whose entries are functions $(a_{ij}(t) = A(t)$, then we can make sense of its derivative; $A'(t) = (a_{ij}'(t))$. Then we have the properties:

$$(A + B)'(t) = A'(t) + B'(t) \text{ , } (AB)'(t) = A'(t)B(t) + A(t)B'(t)$$

Note the order of multiplication in the second formula; it's important!

§3: **Eigenvectors and eigenvalues**

We shall see that our approach to solving $\mathbf{u}' = A\mathbf{u}$, is, like in Chapter 1, to find solutions of the form $\mathbf{u} = e_{kt}\vec{v}_0$ for a (non-zero) vector $\vec{v}_0$ and number k. These vectors and numbers will be determined by A; they will come from solving

$$A\vec{v}_0 = k\vec{v}_0$$

Such vectors are called *eigenvectors*, and their associated numbers k are called *eigenvalues.* Our approach to finding such vectors and numbers is to write the equation as

$$(A - kI)\vec{v}_0 = \vec{0} \qquad (*)$$

First we find the right values k. Linear algebra teaches us that (*) has a (non-zero vector) solution exactly when $\det(A - kI) = 0$, where "det" stands for the determinant. We have already run into this concept; for a 2-by-2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ , } \det(A) = ad - bc$$

There are similar formulas for $n \times n$ matrices for any n, but we will focus on $n = 2$ to make our calculations simpler. If we compute $\det(A - kI)$, we find that (*) has a solution exactly when

$$\det(A - kI) = k^2 - (a + d)k + (ad - bc) = 0$$

This is a quadratic polynomial, so it (usually) has two roots, r_1, r_2 . Once we have the roots, we go back and solve $(A - r_iI)\vec{v} = \vec{(0)}$, i.e,

$$ax_1 + bx_2 = r_ix_1 \text{ , } cx_1 + dx_2 = r_ix_2$$

It turns out that the second equation is always redundant (although this is really only true of 2×2 systems), and we can find $\vec{v}$ by choosing any convenient (non-zero) value for x_1 or x_2, and solving for the other.

§5: Homogeneous linear systems with constant coefficients

Now we put all of this technology to work, to solve our system of differential equations

$$\mathbf{u}' = A\mathbf{u} \qquad (**)$$

where A is a matrix whose entries are all constants. Our basic procedure will be to *guess* that the solution is $\mathbf{u} = e^{kt}\vec{v}_0$ for some vector (with constant entries) $\vec{v}_0$. If we plug such a function into (**), we find that

$$\mathbf{u}' = k\mathbf{u} = e^{kt}(k\vec{v}_0), \text{ while } A\mathbf{u} = e^{kt}(A\vec{v}_0)$$

and so we find that we need $A\vec{v}_0 = k\vec{v}_0$. In other words, we need the eigenvalues for A, and their corresponding eigenvectors. This gives us **two** solutions to the system of equations, but using the Principle of Superposition:

If u_1, u_2 are solutions to the linear system of equations $\mathbf{u}' = A\mathbf{u}$, then the functions $c_1u_1 + c_2u_2$ are also solutions, for any constants c_1, c_2 .

(which we can easily verify), we can take linear combinations of our solutions, to get the general solution to the system of equations

$$\mathbf{u} = c_1e^{r_1t}\vec{v_1} + c_2e^{r_2t}\vec{v_2}$$

where $\vec{v_1}, \vec{v_2}$ are eigenvectors for the coefficient matrix A, with eigenvalues r_1, r_2.

To solve an initial value problem, we find the general solution, and then plug t_0 into the solution and set the vector equal to our initial values $(u_1(t_0), u_2(t_0))$. This gives us a pair of linear equations to solve, which we do using our earlier techniques.

Each solution is a pair of functions $\mathbf{u} = (u_1(t), u_2(t))$, which, if we think of them as giving x- and y-coordinates, describe a *path* in the plane. $\mathbf{u}'$ can then be interpreted as the velocity vectors of this path, which are vectors *tangent* to the path. since we have a solution to our differential equation (**), $\mathbf{u}'$ is actually $A\mathbf{u}$, which we can imagine computing for *every* point in the plane. Plotting each of these vectors $A\vec{v}$ with its tail at $\vec{v}$ gives us a vector field in the plane, which we call the *direction field* of the system of equations. The solutions to (**) are the paths whose velocity vectors are equal to this direction field (and in particular is tanget to the vector field). A picture of the direction field, together with a representative collection of solution curves, is called a *phase portrait* for the system of equations.

Our fundamental solutions $u_i = e^{r_it}\vec{v_i}$ give very special solution curves; the y-coordinate is a (constant) multiple of the x-coordinate, so it parametrizes a straight line out from the origin. With two eigenvalues, we have two straight line solutions to the system of equations. Every other solution will be curved (*actually, this isn't quite true; things are much different if one of the eigenvalues is 0; then *every* solution is a straight line (check it out!)*). We can understand the other solutions in terms of their behavior as $t \to \infty$ and $t \to -\infty$.

We have two lines L_1 and L_2 coming from the eigenvalues r_1 and r_2. If $r_1 > r_2$, then as $t \to \infty$, $e^{r_1 t}/e^{r_2 t} = e^{(r_1 - r_2)t} \to \infty$ and so in any solution

$$c_1 e^{r_1 t}\vec{v_1} + c_2 e^{r_2 t}\vec{v_2}, \text{ the } c_1 e^{r_1 t}\vec{v_1} \text{ term will 'dominate'}$$

i.e., the solution will turn parallel to $\vec{v_1}$. A similar argument shows that as $t \to -\infty$, the solutions will turn parallel to $\vec{v_2}$.

The shape of the solutions also depend on the signs of the eigenvalues. If they are both negative, every solution tends toward the origin as $t \to \infty$, and head to ∞ in the other direction. If one is positive and one negative, then the solutions (other than the straight line ones) tend to ∞ in both directions. If both are positive, then every solution tends to ∞ as $t \to \infty$, and heads to the origin in the other direction.

All of this analysis has the *assumption* that the eigenvalues for our matrix A are *distinct real numbers*. We have yet to deal with the other two possibilities: the eigenvalues are complex (conjugates), or are equal.

§5: **Complex eigenvalues**

To deal with complex eigenvalues $\alpha \pm \beta i$, for example

$$A = \begin{pmatrix} 1 & -4 \\ 3 & -3 \end{pmatrix} \text{ , which has eigenvalues } 1 \pm 3i$$

we do what we did for second order equation. We just *assume* that the solution is

$$\mathbf{u} = e^{(\alpha+\beta i)t}\vec{v}_0 = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\vec{v}$$

To find $\vec{v}$, we solve $A\vec{v} = (\alpha + \beta i)\vec{v}$ as before, except that in this case the coordinates of $\vec{v}$ will be complex numbers. (As before, the second equation is redundant.) If we write

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ so } \mathbf{u} = \begin{pmatrix} (\cos(\beta t) + i\sin(\beta t))\vec{v_1} \\ (\cos(\beta t) + i\sin(\beta t))\vec{v_2} \end{pmatrix}$$

we can write this as

$$\mathbf{u} = \mathbf{x} + i\mathbf{y} \text{ , where } \mathbf{x} \text{ and } \mathbf{y} \text{ have real entries}$$

Then we use the useful fact: if $\mathbf{u} = \mathbf{x} + i\mathbf{y}$ solves the equation $\mathbf{u}' = A\mathbf{u}$, where A has real entries, then $\mathbf{x}$ and $\mathbf{y}$ *also* solve the system of equations. (This uses the fact that eigenvalues come in complex conjugate pairs!) These two vector functions are our fundamental solutions. Each coordinate of these functions is a linear combination of the functions $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$, and so the phase portrait of such a system of equations invloves both a circular motion and and expension from or contraction towards the origin (depending on whether α is positive or negative). The solution curves are spirals around the origin.

§5: **Repeated eigenvalues**

If the coefficient matrix has only one eigenvalue r, occuring twice, then using that eigenvalue we can find an eigenvector $\vec{v}$ and a solition $\mathbf{u} = e^{rt}\vec{v}$. But only if $A = rI$ (in the 2×2 case, will we be able to find *two* independent solutions; in that case we can actually take

$$\mathbf{u} = c_1 r^{rt}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{rt}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{rt} \\ c_2 e^{rt} \end{pmatrix}$$

In every other case, we *guess* that the other solution is $\mathbf{u} = te^{rt}\vec{w}$. It turns out, however, that this won't work: what we instead need to do is to guess

$\mathbf{u} = te^{rt}\vec{v} + e^{rt}\vec{w}$, where v is the *same* eigenvector with eigenvalue r that we already found!

Carrying this expression through the equation $\mathbf{u}' = A\mathbf{u}$, we find that $\vec{w}$ must satisfy

$$(A - rI)\vec{w} = \vec{v}$$

Since we know r and $\vec{v}$, we can solve this for $\vec{w}$. (In general, linear algebra tells us we shouldn't always expect to be able to solve such an equation, but because of the repeated root, it turns out than in fact we *can*.) This gives us our second fundamental solution.

If we look at the phase portrait for such an equation, we find that for the solution

$$\mathbf{u} = c_1 e^{rt}\vec{v} + c_2(te^{rt}\vec{v} + e^{rt}\vec{w}) = e^{rt}[(c_1 + c_2 t)\vec{v} + c_2\vec{w}]$$

as $t \to \infty$ the solution curves (with $c_2 > 0$) run parallel to $\vec{v}$, while as $t \to -\infty$, they run parallel to $-\vec{v}$, that is, the term $t\vec{v}$ will dominate. If $c_2 < 0$, the roles of $\pm\vec{v}$ are reversed. Notice that $c_2 = 0$ gives the straight line solution(s) $\mathbf{u} = \pm e^{rt}\vec{v}$. The solutions will tend to the origin as t goes to ∞ or $-\infty$, depending on the sign of r.