Chapter 2: Limits and Continuity

Rates of change and limits:
Limit of a function $f$ at a point $a$ = the value the function ‘should’ take at the point $a$
= the value that the points ‘near’ $a$ tell you $f$ should have at $a$
$\lim_{x \to a} f(x) = L$ means $f(x)$ is close to $L$ when $x$ is close to (but not equal to) $a$

Idea: slopes of tangent lines

$\lim_{x \to a} f(x) = L$ does not care what $f(a)$ is; it ignores it
$\lim_{x \to a} f(x)$ need not exist! (function can’t make up it’s mind?)

Rules for finding limits:
If two functions $f(x)$ and $g(x)$ agree (are equal) for every $x$ near $a$
(but maybe not at $a$), then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$

Ex.: $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 4} = \lim_{x \to 2} \frac{(x-1)(x-2)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x-1}{x+2} = 1/4$

If $f(x) \to L$ and $g(x) \to M$ as $x \to a$ (and $c$ is a constant), then
$f(x) + g(x) \to L + M$; $f(x) - g(x) \to L - M$; $cf(x) \to cL$
$f(x)g(x) \to LM$; and $f(x)/g(x) \to L/M$ \textit{provided} $M \neq 0$

If $f(x)$ is a polynomial, then $\lim_{x \to x_0} f(x) = f(x_0)$

Basic principle: to solve $\lim_{x \to x_0}$, plug in $x = x_0$!
If (and when) you get $0/0$, try something else! (Factor $(x-a)$ out of top and bottom...)
If a function has something like $\sqrt{x} - \sqrt{a}$ in it, try multiplying (top and bottom)
with $\sqrt{x} + \sqrt{a}$
(idea: $u = \sqrt{x}, v = \sqrt{a}$, then $x - a = u^2 - v^2 = (u - v)(u + v)$.)
Sandwich Theorem: If \(f(x) \leq g(x) \leq h(x)\), for all \(x\) near \(a\) (but not at \(a\)), and
\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \text{ then } \lim_{x \to a} g(x) = L.
\]

One-sided limits:
Motivation: the Heaviside function

\[y = H(x)\]

The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, \(H(0)\) 'wants' to be 0, while on the right, \(H(0)\) 'wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the 'one-sided' limits DO exist.

Limit from the right: \(\lim_{x \to a^+} f(x) = L\) means \(f(x)\) is close to \(L\) when \(x\) is close to, and bigger than, \(a\)

Limit from the left: \(\lim_{x \to a^-} f(x) = M\) means \(f(x)\) is close to \(M\) when \(x\) is close to, and smaller than, \(a\)

\[\lim_{x \to a} f(x) = L\] then means \(\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L\)
(i.e., both one-sided limits exist, and are equal)

Limits at infinity / infinite limits:
\(\infty\) represents something bigger than any number we can think of.

\[\lim_{x \to \infty} f(x) = L\] means \(f(x)\) is close of \(L\) when \(x\) is really large.

\[\lim_{x \to -\infty} f(x) = M\] means \(f(x)\) is close of \(M\) when \(x\) is really large and negative.

Basic fact: \(\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0\)

More complicated functions: divide by the highest power of \(x\) in the denominator.
\(f(x), g(x)\) polynomials, degree of \(f = n\), degree of \(g = m\)

\[\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = 0\] if \(n < m\)

\[\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = \frac{\text{coeff of highest power in } f}{\text{coeff of highest power in } g}\] if \(n = m\)

\[\lim_{x \to \pm \infty} \frac{f(x)}{g(x)} = \pm \infty\] if \(n > m\)

\[\lim_{x \to a} f(x) = \infty\] means \(f(x)\) gets really large as \(x\) gets close to \(a\)
Also have \( \lim_{x \to a^-} f(x) = -\infty \); \( \lim_{x \to a^+} f(x) = \infty \); \( \lim_{x \to a^-} f(x) = \infty \); etc.

Typically, an infinite limit occurs where the denominator of \( f(x) \) is zero (although not always)

**Asymptotes:**

The line \( y = a \) is a horizontal asymptote for a function \( f \) if

\[
\lim_{x \to \infty} f(x) \text{ or } \lim_{x \to -\infty} f(x) \text{ is equal to } a.
\]

I.e., the graph of \( f \) gets really close to \( y = a \) as \( x \to \infty \) or \( a \to -\infty \)

The line \( x = b \) is a vertical asymptote for \( f \) if \( f \to \pm \infty \) as \( x \to b \) from the right or left.

If numerator and denominator of a rational function have no common roots, then vertical asymptotes = roots of denom.

**Continuity:**

A function \( f \) is continuous (cts) at \( a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

This means: (1) \( \lim_{x \to a} f(x) \) exists; (2) \( f(a) \) exists; and (3) they’re equal.

Limit theorems say (sum, difference, product, quotient) of cts functions are cts.

Polynomials are continuous at every point;

rational functions are continuous except where denom=0.

Points where a function is not continuous are called discontinuities

Four flavors:

- removable: both one-sided limits are the same
- jump: one-sided limits exist, not the same
- infinite: one or both one-sided limits is \( \infty \) or \( -\infty \)
- oscillating: one or both one-sided limits DNE

**Intermediate Value Theorem:**

If \( f(x) \) is cts at every point in an interval \([a, b]\), and \( M \) is between \( f(a) \) and \( f(b) \),

then there is (at least one) \( c \) between \( a \) and \( b \) so that \( f(c) = M \).

**Application:** finding roots of polynomials

**Tangent lines:**

Slope of tangent line = limit of slopes of secant lines; at \( (a, f(a)) \):

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Notation: call this limit \( f'(a) = \text{derivative of } f \text{ at } a \)

Different formulation: \( h = x - a, x = a + h \)

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \text{limit of difference quotient}
\]

If \( y = f(x) \) = position at ‘time’ \( x \), then difference quotient = average velocity;
limit = instantaneous velocity.
Chapter 3: Derivatives

The derivative of a function:

\[ \text{derivative} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

If \( f'(a) \) exists, we say \( f \) is differentiable at \( a \)

Fact: \( f \) differentiable (diff’ble) at \( a \), then \( f \) cts at \( a \)

Using \( h \to 0 \) notation: replace \( a \) with \( x \) (= variable), get \( f'(x) = \text{new function} \)

Or:

\[ f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z-x} \]

\( f'(x) \) = the derivative of \( f = \text{function whose values are the slopes of the tangent lines to the graph of } y = f(x). \) Domain = every point where the limit exists

Notation:

\[ f'(x) = \frac{dy}{dx} = \frac{df}{dx} = y' = D_x f = D f = (f(x))' \]

Differentiation rules:

\[
\frac{d}{dx} (\text{constant}) = 0 \quad \frac{d}{dx} (x) = 1
\]

\[
(f(x)+g(x))' = f'(x) + g'(x) \quad (f(x)-g(x))' = f'(x) - g'(x) \\
(c f(x))' = c (f(x))'
\]

\[ (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \]

\[
(x^n)' = nx^{n-1} , \quad \text{for } n = \text{natural number} \quad a^x \quad (a^x)' = K \cdot a^x, \text{where } K = \left. \frac{d}{dx} (a^x) \right|_{x=0} \\
\left[ (1/g(x))' = -g'(x)/(g(x))^2 \quad \right]
\]

Higher derivatives:

\( f'(x) \) is ‘just’ a function, so we can take its derivative!

\[ (f'(x))' = f''(x) \quad (= y'') = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} \]

= second derivative of \( f \)

Keep going! \( f'''(x) = 3\text{rd derivative}, f^{(n)}(x) = n\text{th derivative} \)

Rates of change

Physical interpretation:

\( f(t) = \text{position at time } t \)

\( f'(t) = \text{rate of change of position} = \text{velocity} \)

\( f''(t) = \text{rate of change of velocity} = \text{acceleration} \)

\( |f'(t)| = \text{speed} \)

Basic principle: for object to change direction (velocity changes sign),

\( f'(t) = 0 \) somewhere (IVT!)

Examples:

Free-fall: object falling near earth; \( s(t) = s_0 + v_0 t - \frac{g}{2} t^2 \)

\( s_0 = s(0) = \text{initial position}; v_0 = \text{initial velocity}; g = \text{acceleration due to gravity} \)
Economics:

\( C(x) = \text{cost of making } x \text{ objects}; \ R(x) = \text{revenue from selling } x \text{ objects}; \)

\( P = R - C = \text{profit} \)

\( C'(x) = \text{marginal cost} = \text{cost of making ‘one more’ object} \)

\( R'(x) = \text{marginal revenue}; \text{ profit is maximized when } P'(x) = 0; \)

i.e., \( R'(x) = C'(x) \)

**Derivatives of trigonometric functions**

Basic limit: \( \lim_{x \to 0} \frac{\sin x}{x} = 1; \) everything else comes from this!

\( \lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \)

Note: this uses radian measure!

\( \lim_{x \to 0} \frac{\sin(bx)}{x} = \lim_{x \to 0} \frac{b\sin(bx)}{bx} = \lim_{u \to 0} \frac{b\sin(u)}{u} = b \)

Then we get:

\( (\sin x)' = \cos x \)

\( (\cos x)' = -\sin x \)

\( (\tan x)' = \sec^2 x \)

\( (\cot x)' = -\csc^2 x \)

\( (\sec x)' = \sec x \tan x \)

\( (\csc x)' = -\csc x \cot x \)