Math 221 - Section 5

A quick guide to homogeneous linear systems of DEs

A *standard form* homogeneous linear system of two first order constant coefficient differential equations is

$$\frac{dx}{dt} = ax + by \qquad\qquad \frac{dy}{dt} = cx + dy$$

where a, b, c, and d are constants. Both x and y are dependent variables, depending on the independent variable t. The *solution* to this system is a pair of functions $x(t)$ and $y(t)$ that satisfy both DEs together. Often, it will be convenient to write the solution as a column vector

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

To solve this system when $b \neq 0$, we use the first DE to *eliminate* the variable y from the second DE. Solving the first DE for y:

$$\boxed{y = \frac{1}{b}(x' - ax) \ .}$$

Then the derivative

$$y' = \frac{1}{b}x'' - \frac{a}{b}x' \ .$$

Plugging these into the second DE of the system gives

$$\frac{1}{b}x'' - \frac{a}{b}x' = cx + d(\frac{1}{b}(x' - ax)) \ .$$

Rearranging the terms of this equation and multiplying both sides by b, we now have one second order linear constant coefficient DE to solve for x, namely

$$\boxed{x'' - (a+d)x' + (ad - bc)x = 0 \ .}$$

So we solve this DE in the usual way (chap. 3 sec. 3 of the text). Once we have x, we can use the formula in the first box to find y.

Using the vocabulary discussed in chapter 5 of the text, the characteristic roots r_1, r_2 from the auxiliary equation for the constant coefficient second order homogeneous DE above for x are also the *eigenvalues of the matrix*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

An *eigenvector* associated to the root r_i is the vector $\begin{bmatrix} 1 \\ \frac{r_i - a}{b} \end{bmatrix}$. These vectors are discussed further in examples below.

Homogeneous linear systems always have a critical point at $(0, 0)$; that is, the constant functions $x(t) \equiv 0$ and $y(t) \equiv 0$ are solutions of this system for any values of a, b, c, and d. The behavior around the critical point (0,0) depends on the characteristic roots of the auxiliary equation. For example, the critical point can be stable, unstable (a source), or asymptotically stable (a sink). The critical point is described in various ways as a node, saddle, spiral, center, or star. See the pictures on the last page of this handout for pictorial definitions of these terms.

Example (A): Find the general solution to the linear system

$$\frac{dx}{dt} = 0x - 2y \ , \qquad \frac{dy}{dt} = x + 3y \ ,$$

and analyze the phase plane direction field for this system.

Answer: Here we have $b = -2 \neq 0$ in the first DE, we can solve for y and get $y = -\frac{1}{2}x'$. Plugging this into the second DE, then $-\frac{1}{2}x'' = x + 3(-\frac{1}{2}x')$. Then the associated second order equation (also from the boxed formula on p. 1) is $x'' - 3x' + 2x = 0$. This has roots $1, 2$, so the solution for x is

$$x(t) = C_1 e^t + C_2 e^{2t}.$$

Using the formula (above or from the boxes) for $y(t)$, we get $y = -\frac{1}{2}x' = -\frac{1}{2}C_1 e^t - C_2 e^{2t}$.

The general solution for the system is

$$x(t) = C_1 e^t + C_2 e^{2t} \qquad\qquad y(t) = -\frac{1}{2}C_1 e^t - C_2 e^{2t}\ .$$

Written as a vector, the general solution is $\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(*Note :* It is important to notice that the same constants appear in the solutions for both $x(t)$ and $y(t)$, so we cannot replace a multiple of an arbitrary constant with an arbitrary constant in only one of the solutions.)

The x-nullcline is the line $y = 0$; arrows point to the right when $y < 0$ and to the left when $y > 0$. The y-nullcline is the line $y = -\frac{1}{3}x$; arrows point upward when $y > -\frac{1}{3}x$ and downward when $y < -\frac{1}{3}x$.

The solution with $C_1 = 1$ and $C_2 = 0$ is given by $x(t) = e^t, y(t) = -\frac{1}{2}e^t$; then for all t, this solution follows the ray $y = -\frac{1}{2}x$ for $x > 0$ and $y < 0$ in the phase plane, going up and to the right in the direction of the eigenvector $\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ for the eigenvalue/root $r_1 = 1$ as t increases. The solution with $C_1 = 0$ and $C_2 = 1$ has $x(t) = e^{2t}, y(t) = -e^{2t}$, and so follows the ray $y = -x$ with $x > 0$ and $y < 0$ in the direction of the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for the eigenvalue/root $r_2 = 2$ as $t \to \infty$.

For the solution of the IVP with $x(0) = 12$ and $y(0) = -5$, as $t \to \infty$ we have $x(t) \to -\infty$ and $y(t) \to \infty$. However, for this IVP the solution $(x(t), y(t))$ can never reach the point $(-7, 6)$.

The critical point $(0, 0)$ is unstable; the characteristic roots 1,2 for x distinct positive real numbers, and so the critical point is a node source.

Example (B): Find the general solution of the system

$$\frac{dx}{dt} = x + 2y\ , \qquad \frac{dy}{dt} = -5x - y\ ,$$

and analyze stability of the critical point (0,0).

Answer: We have $b = 2 \neq 0$ in the first DE, we can solve for y and get $y = \frac{1}{2}x' - \frac{1}{2}x$. Plugging this into the second DE, then $\frac{1}{2}x'' - \frac{1}{2}x' = -5x - (\frac{1}{2}x' - \frac{1}{2}x)$. Then the associated second order equation is $x'' + 9x = 0$. This has roots $\pm 3i$, so the solution for x is

$$x(t) = C_1 \sin(3t) + C_2 \cos(3t).$$

Using the formula for $y(t)$, we get

$$\begin{aligned} y(t) &= (1/2)(3C_1 \cos(3t) - 3C_2 \sin(3t)) - (1/2)(C_1 \sin(3t) + C_2 \cos(3t)) \\ &= C_1 \left((3/2)\cos(3t) - (1/2)\sin(3t)\right) + C_2 \left(-(1/2)\cos(3t) - (3/2)\sin(3t)\right). \end{aligned}$$

Written as a vector, the general solution is

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} \sin(3t) \\ (3/2)\cos(3t) - (1/2)\sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(3t) \\ -(1/2)\cos(3t) - (3/2)\sin(3t) \end{bmatrix}.$$

This general solution involves only cosine and sine functions, but not exponential functions, so $x(t)$ and $y(t)$ can't (asymptotically) approach 0, and they also can't go to $\pm\infty$. The critical point (0,0) for this linear system is a stable center.

Example (C): Two large tanks are interconnected by pipes. Fresh water flows into tank A at a rate of 6 L/min. Solution flows through a pipe from tank A to tank B at a rate of 8 L/min, and solution flows through another pipe from tank B to tank A at a rate of 2 L/min. The solution in tank B also drains out onto the ground at a rate of 6 L/min. The solutions in both tanks are kept evenly mixed. Initially tank A contains 3 g of salt dissolved in 24 L of water, and tank B contains 5 g of salt dissolved in 24 L of water. Set up an initial value problem modeling this situation.

Answer: Let t be the time in minutes, with initial time $t = 0$. Let $x(t)$ be the mass of salt in grams in tank A at time t, and let $y(t)$ be the mass of salt in grams in tank B at time t. The initial conditions say that $x(0) = 3$ and $y(0) = 5$.

The volume of solution in tank A is changing at a rate of $6 - 8 + 2 = 0$ L/min, so the volume in that tank is a constant 24 L. The volume in tank B changes at a rate of $8 - 2 - 6 = 0$ L/min, so the volume in that tank is also a constant 24 L/min.

First analyze the input and output of salt at tank A. The fresh water flowing into tank A contains 0 g/L of water, so the input from this pipe changes the mass of salt in tank A at a rate of (0 g/L)(6 L/min)=0 g/min. The solution flowing from tank A to tank B removes salt from tank A at a rate of $(\frac{x(t)}{24}$ g/L)(8 L/min). The solution flowing from tank B to tank A inputs salt into tank A at a rate of $(\frac{y(t)}{24}$ g/L)(2 L/min). Since $\frac{dx}{dt}$ = (input rate) - (output rate), then $\frac{dx}{dt} = 0 + \frac{2y}{24} - \frac{8x}{24}$ g/min.

Next do the same for tank B: $\frac{dy}{dt}$ = (input rate) - (output rate) = $\frac{8x}{24} - \frac{2y}{24} - \frac{6y}{24}$ g/min.

The initial value problem modeling this situation is:

$$\frac{dx}{dt} = -\frac{1}{3}x + \frac{1}{12}y\ , \qquad \frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y\ ; \qquad x(0) = 3\ , \qquad y(0) = 5\ .$$

Example (D): Solve the initial value problem from Example (C), and discuss what happens to the solution as $t \to \infty$.

Answer. Here we have $a = -\frac{1}{3}$, $b = \frac{1}{12}$, $c = \frac{1}{3}$, and $d = -\frac{1}{3}$, with $b \neq 0$. The associated second order DE is $x'' - (-\frac{1}{3} - \frac{1}{3})x' + ((-\frac{1}{3})(-\frac{1}{3}) - (\frac{1}{12})(\frac{1}{3}))x = 0$, or $x'' + \frac{2}{3}x' + \frac{1}{12}x = 0$. This DE has auxiliary equation $r^2 + \frac{2}{3}r + \frac{1}{12} = 0$, with roots $r_1 = -\frac{1}{6}$ and $r_2 = -\frac{1}{2}$. Then the solution for x is

$$x(t) = C_1 e^{-(1/6)t} + C_2 e^{-(1/2)t}.$$

Then

$$\begin{aligned} y &= (1/\tfrac{1}{12})(x' - (-\tfrac{1}{3})x) = 12x' + 4x \\ &= 12(-\tfrac{1}{6}C_1 e^{-(1/6)t} - \tfrac{1}{2}C_2 e^{-(1/2)t}) + 4(C_1 e^{-(1/6)t} + C_2 e^{-(1/2)t}) \\ &= 2C_1 e^{-(1/6)t} - 2C_2 e^{-(1/2)t} \end{aligned}$$

Plugging the initial values into this general solution gives

$$x(0) = C_1 + C_2 = 3 \qquad y(0) = 2C_1 - 2C_2 = 5.$$

These equations have solution $C_1 = \frac{11}{4}$ and $C_2 = \frac{1}{4}$. Then the solution of the initial value problem is

$$x(t) = \frac{11}{4}e^{-(1/6)t} + \frac{1}{4}e^{-(1/2)t} \ , \qquad y(t) = \frac{11}{2}e^{-(1/6)t} - \frac{1}{2}e^{-(1/2)t} \ .$$

As $t \to \infty$, the functions $x(t)$ and $y(t)$ in these solutions (asymptotically) approach 0. (The critical point (0,0) for this linear system is a node sink.)

Exercises:

(1) Find the general solution to the linear system if b is zero. *Hint:* The first DE has no y's, so solve it first and plug that solution into the second DE.

For exercises (2)-(6), find the general solution (or IVP solution when initial values are given), draw the phase plane direction field, and discuss the stability of the critical point (0,0):

(2)=(4.1.15) $\dfrac{dx}{dt} = \dfrac{1}{2}y \ , \qquad \dfrac{dy}{dt} = -8x \ .$

(3)=(4.2.3) $\dfrac{dx}{dt} = -3x + 2y \ , \qquad \dfrac{dy}{dt} = -3x + 4y \ ; \qquad x(0) = 0 \ , \qquad y(0) = 2 \ .$

(4)=(4.2.6) $\dfrac{dx}{dt} = x + 9y \ , \qquad \dfrac{dy}{dt} = -2x - 5y \ ; \qquad x(0) = 3 \ , \qquad y(0) = 2 \ .$

(5)=(6.1.18) $\dfrac{dx}{dt} = -y \ , \qquad \dfrac{dy}{dt} = 4x \ .$

(6)=(6.1.20) $\dfrac{dx}{dt} = y \ , \qquad \dfrac{dy}{dt} = -5x - 4y \ .$

Exercises (7)-(8) deal with a two-tank system. Fresh water flows into tank A, and brine flows from tank B onto the ground, both at a rate of $2r$ L/min. Brine flows from tank B to tank A at a rate of r L/min, and brine flows from tank A to tank B at a rate of $3r$ L/min. The initial amounts $x(0)$ and $y(0)$ of salt in tanks A and B, respectively, are given, as are the volumes V_A and V_B in the tanks. Set up and solve the initial value problem modeling the situation in each exercise.

(7)=(4.2.30) $r = 10$ L/min $\quad x(0) = 0.5$ g, $\quad y(0) = 0.5$ g, $\quad V_A = 100$ L, $\quad V_B = 200$ L.

(8) $r = 3$ L/min $\quad x(0) = 1$ g, $\quad y(0) = 2$ g, $\quad V_A = 20$ L, $\quad V_B = 20$ L.