Math 221 - Section 5

A quick guide to sketching phase planes

Our text discusses equilibrium points and analysis of the phase plane. However, there is one idea, not mentioned in the book, that is very useful to sketching and analyzing phase planes, namely *nullclines*. Recall the basic setup for an autonomous system of two DEs:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

To sketch the phase plane of such a system, at each point (x_0, y_0) in the xy-plane, we draw a vector starting at (x_0, y_0) in the direction $f(x_0, y_0)\mathbf{i} + g(x_0, y_0)\mathbf{j}$.

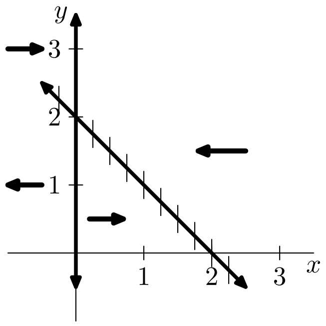
Definition of nullcline. The x**-nullcline** is a set of points in the phase plane so that $\frac{dx}{dt} = 0$. Geometrically, these are the points where the vectors are either straight up or straight down. Algebraically, we find the x-nullcline by solving $f(x, y) = 0$.

The y**-nullcline** is a set of points in the phase plane so that $\frac{dy}{dt} = 0$. Geometrically, these are the points where the vectors are horizontal, going either to the left or to the right. Algebraically, we find the y-nullcline by solving $g(x, y) = 0$.

How to use nullclines. Consider the system

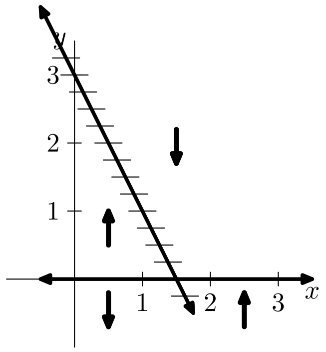
$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy,$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

To find the x-nullcline, we solve $2x\left(1 - \frac{x}{2}\right) - xy = 0$, where multiplying out and collecting the common factor of x gives $x(2-x-y) = 0$. This gives two x-nullclines, the line $x+y = 2$ and the y-axis. Solutions of this system move to the right if $x(2-x-y) > 0$, so the direction field arrows will point toward the right (and either upward or downward) in two cases: When both $x > 0$ and $(2-x-y) > 0$ (to the right of the y axis and below the line $y = -x+2$), and when both $x < 0$ and $(2-x-y) < 0$ (to the left of the y axis and above the line $y = -x+2$).

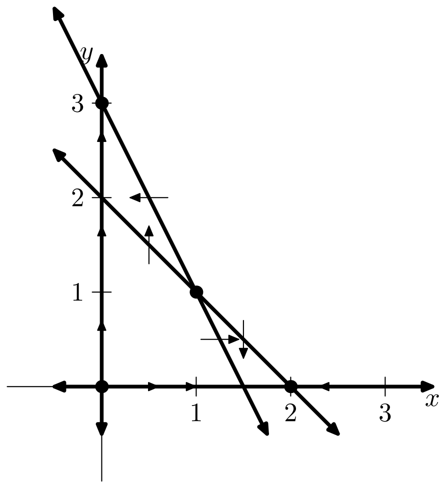


To find the y-nullcline, we solve $3y\left(1 - \frac{y}{3}\right) - 2xy = 0$, where multiplying out and collecting the common factor of y gives $y(3-y-2x) = 0$. This gives two y-nullclines, the line $2x+y = 3$ and the x-axis. Solutions of this system move upward if $y(3-y-2x) > 0$, so the direction field arrows will point up (and either right or left) in two cases: When both $y > 0$ and $(3-y-2x) > 0$

(above the x axis and below the line $y = -2x + 3$), and when both $y < 0$ and $(3 - y - 2x) < 0$ (below the x axis and above the line $y = -2x + 3$).



Combining this information gives us the following picture. Notice that we can draw directions on each nullcline by using the direction information from the other graph. For example, the line segment from $(1, 1)$ to $(0, 3)$, since it is above the line $y = -x + 2$ and to the right of the y-axis, has solutions moving to the left.



Also, where the x-nullcline and y-nullcline cross, both $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are zero. So these points (marked by dots in the above graph) are equilibrium points.

Once a solution enters the triangle with vertices $(1, 1)$, $(0, 2)$ and $(0, 3)$, it can never leave. Similarly, solutions in the triangle with vertices $(1, 1)$, $(3/2, 0)$ and $(2, 0)$ can never leave.

Exercises. Graph the nullclines, sketch the direction fields, and discuss the possible fates of solutions for the following systems. Note: The nullclines may not be straight lines.

(1) $\dfrac{dx}{dt} = x(-x - 3y + 150), \quad \dfrac{dy}{dt} = y(-2x - y + 100).$

(2) $\dfrac{dx}{dt} = x(10 - x - y), \quad \dfrac{dy}{dt} = y(30 - 2x - y).$

(3) $\dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{2}\right) - xy, \quad \dfrac{dy}{dt} = y\left(\dfrac{9}{4} - y^2\right) - x^2 y.$

(4) $\dfrac{dx}{dt} = x(-4x - y + 160), \quad \dfrac{dy}{dt} = y(-x^2 - y^2 + 2500).$