Math 221

Topics for the second exam

Don't forget the topics for the first exam!

Basic object of study: **second (and higher) order linear differential equations**

$$(\blacktriangledown) \qquad y'' + p(t)y' + q(t)y = g(t)$$

Initial value problem:

$$y(t_0){=}y_0 \qquad \text{and} \qquad y'(t_0){=}y'_0$$

Basic fact: if $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval around t_0, then any initial value problem has a *unique* solution on that interval. Our Basic goal: find the solution!
Homogeneous: $g(t) = 0$ Constant coefficients: $p(t)$ and $q(t)$ are constant.

Operator notation: write $L[y] = y'' + p(t)y' + q(t)y$ (this is called a *linear operator*), then a solution to ($\blacktriangledown$) is a function y with $L[y] = g(t)$.
For a linear differential equation, $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[yt]$, and so if y_1 and y_2 are both solutions to $L[y] = 0$ then so is $c_1y_1 + c_2y_2$. $c_1y_1 + c_2y_2$ is called a *linear combination* of y_1 and y_2. This is called the *Principle of Superposition*: more generally, if $L[y_1] = g_1(t)$ and $L[y_2] = g_2(t)$, then $L[y_1 + y_2] = g_1(t) + g_2(t)$.

With (the right) *two* solutions y_1, y_2 to a *homogeneous* equation

$$(\blacktriangledown\blacktriangledown) \qquad y'' + p(t)y' + q(t)y = 0$$

we can solve *any* initial value problem, by choosing the right linear combination: we need to solve

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y'_1(t_0) + c_2y'_2(t_0) = y'_0$$

for the constants c_1 amd c_2; then $y = c_1y_1 + c_2y_2$ is our solution. This we can do directly, as a pair of linear equations, by solving one equation for one of the constants, and plugging into the other equation, or we can use the formulas

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. This makes it clear that a solution *exists* (i.e., we have the 'right' pair of functions), provided that the quantity

$$W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} \neq 0$$

W is called the *Wronskian* (determinant) of y_1 and y_2 at t_0 . The Wronskian is closely related to the concept of *linear independence* of a collection y_1,...,y_n of functions; such a collection is linearly independent if the only linear combination $c_1y_1 + \cdots + c_ny_n$ which is equal to the 0 function is the one with $c_1 = \cdots = c_n = 0$.

Two functions y_1 and y_2 are linearly independent if their Wronksian is non-zero at *some* point; for a pair of solutions to ($\blacktriangledown\blacktriangledown$), it turns out that the Wronskian is always equal to a constant multiple of

$$exp(-\int p(t)\ dt)$$

and so is either *always* 0 or *never* 0. We call a pair of linearly independent solutions to (▼▼) a pair of *fundamental solutions*. By our above discussion, we can solve any initial value problem for (▼▼) as a linear combination of fundamental solutions y_1 and y_2. By our existence and uniqueness result, this give us:

If y_1 and y_2 are a fundamental set of solutions to the differential equation (▼▼), then *any* solution to (▼▼) can be expressed as a linear combination $c_1y_1 + c_2y_2$ of y_1 and y_2.

So to solve an initial value problem for (▼▼), all we need is a pair of fundamental solutions.

Homogenous equations with constant coefficients: $ay'' + by' + cy = 0$

Basic idea: *guess* that $y = e^{rt}$, and plug in! Get:

$$(ar^2 + br + c)e^{rt} = 0 \quad , \text{so} \quad ar^2 + br + c = 0$$

Solve: get (typically) two roots r_1, r_2, so $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$ are *both* solutions.
The equation $ar^2 + br + c = 0$ is called the *auxiliary equation* for our differential equation.

If the roots of the characteristic equation are real and distinct, $r_1 \neq r_2$, then a fundamental set of solutions is

$$y_1 = e^{r_1t}, y_2 = e^{r_2t}$$

If the root of the characteristic equation are complex $\alpha \pm \beta i$, then a fundamental set of solutions is

$$y_1 = e^{\alpha t}\cos(\beta t), y_2 = e^{\alpha t}\sin(\beta t)$$

If the roots of the characteristic equation are *repeated* (and therefore real), $r_1 = r_2 = r$, then a fundamental set of solutions is

$$y_1 = e^{rt}, y_2 = te^{rt}$$

Reduction of order is a general technique for finding a second, linearly independent, solution y_2 to (▼▼), given a (non-zero) solution y_1; if y_1 is a solution to (▼▼), then so is

$$y_2 = y_1(t)\int \frac{\exp(-\int p(t)\ dt)}{(y_1(t))^2}\ dt$$

This formula was found by *assuming* that $y_2(t) = c(t)y_1(t)$, and then determining what differential equation $c(t)$ must satisfy! It turns out to be a first-order equation (hence the name reduction of order).

Higher order equations: Much of what we just did for second order equations goes through without any change for even higher order (linear) equations:

$$(\blacklozenge) \qquad L[y] = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)$$

and its associated homogeneous equation

$$(\blacklozenge\blacklozenge) \qquad y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0$$

In this case the correct notion of an initial value problem requires us to specify the values, at t_0, of y and all its derivatives up to the $(n-1)$st:

$$y(t_0) = y_0\ ,\ y'(t_0) = y_0'\ ,\ \ldots\ ,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

As with the second order case, we have a principle of superposition: $L[y_1] = g_1$ and $L[y_2] = g_2$, then $L[y_1 + y_2] = g_1 + g_2$. This means that linear combinations of solutions to the homogeneous equation (♦♦) are also solutions. And the general solution to (♦♦) can always be obtained (uniquely) as a linear combination of n *linearly independent* (or *fundamental*) solutions. Linear independence can be determined by computing a Wronskian determinant $W(y_1, \ldots, y_n)$.

The theory we developed for homogeneous equations with constant coefficients can be similarly extended. The equation
$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$$
has a fundamental set of solutions determined by its characteristic equation
$$a_0r^n + \cdots + a_{n-1}r + a_n = 0$$
Real roots r correspond to solutions $\exp(rt)$; complex roots to solutions $\exp(\alpha t)\cos(\beta t)$ and $\exp(\alpha t)\sin(\beta t)$. The only extra wrinkle is that we can have repeated roots which repeat *many* times, and even repeated complex roots! For each, we do as we did before and create new fundamental solutions by multiplying our basic solution by t, as many times as it repeats. For example, the equation
$$y^{(4)} + 2y'' + y = 0$$
has a characteristic equation with roots $i, i, -i$, and $-i$, and so its fundamental solutions are
$$\cos(t), t\cos(t), \sin(t), \text{ and } t\sin(t)$$

Inhomogeneous linear equations: We can solve an inhomogeneous equation
$$(\blacklozenge) \qquad L[y] = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)$$
with $g(t) \neq 0$, by using our knowledge of the solution to its associated homogeneous equation. The principle of superposition tells us that for any pair of solutions Y_1, Y_2 to ($\blacklozenge$), $L[Y_2 - Y_1] = 0$, and so if we have a fundamental set of solutions to the associated homogeneous equation, $y_1, \dots, y_n$, we can write
$$Y_2 = Y_1 + c_1y_1 + \cdots + c_ny_n$$
In other words, we can find *any* solution to ($\blacklozenge$) by finding one *particular* solution, together with a fundamental set of solutions to the associated homogeneous equation ($\blacklozenge\blacklozenge$). Any initial value problem can then be solved by solving the system of equations
$$Y_1(t_0) + c_1y_1(t_0) + \cdots + c_ny_n(t_0) = g(t_0)$$
$$Y_1'(t_0) + c_1y_1'(t_0) + \cdots + c_ny_n'(t_0) = g'(t_0)$$
all the way to
$$Y_1^{(n-1)}(t_0) + c_1y_1^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) = g^{(n-1)}(t_0)$$
for the constants $c_1, \dots, c_n$.

The only part of this we haven't really explored yet is finding a particular solution to ($\blacklozenge$). For this we have two techniques.

Variation of parameters: the idea is to start with a pair of fundamental solutions y_1, yt to the associated homogeneous equation
$$(\blacktriangledown\blacktriangledown) \qquad y'' + p(t)y' + q(t)y = 0$$
and then *guess* that the solution to our inhomogeneous equation
$$(\blacktriangledown) \qquad y'' + p(t)y' + q(t)y = g(t)$$
is of the form $y(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$, and plug in. The resulting equation is too complicated, but if we make the simplifying assumption
$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$$
then the equation becomes
$$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t)$$
which we can solve:

$$c_1' = \frac{-gy_2}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2' = \frac{gy_1}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

Here again, the by now familiar Wronskian appears! Note that we must still *integrate* these functions, to determine c_1 and c_2 .

Method of Undetermined Coefficients: Our second approach to solving inhomogeneous equation involves "educated guessing".

Important: This generally works only for equations with constant coefficients!

The basic idea behind the technique is that for most kinds of functions, like polynomials, expoential, sines and cosines, or products of these, all of the derivatives of the function are of the *same basic form*. So if the function $g(t)$ in

$$(\blacktriangleright) \qquad L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = g(t)$$

is one of these kinds, what we do is *guess* that our solution y is the same kind. We include *undetermined coefficients* in the solution (hence the name), and by plugging into the differential equation and setting equal to our target function, we *solve* for the undetermined coefficents. In particular,

If g is a polynomial of degree n, we set y to be a (different) polynomial of degree n,

If g is a multiple of an exponential $\exp(rt)$, we set y to be a a multiple $c\exp(rt)$ of g,

If g is a multiple of $\sin(\beta t)$ or $\cos(\beta t)$, we set y to be a linear combination $a\sin(\beta t) + b\cos(\beta t)$,

If $g(t) = \exp(rt)\cos(\beta t)$ (or has a sine), we set y to be $a\exp(rt)\sin(\beta t)+b\exp(rt)\cos(\beta t)$,

If g is a polynomial of degree n times one of these, we set y to be a (different, unknown) polynomial of degree n times the corresponding function above.

Then we must plug this function into ($\blacktriangleright$), and solve for the undetermined coefficients.

Of course, there is one wrinkle; sometimes our choice of y *cannot* work, because it is a solution to the associated *homogeneous* equation. For example, for the equation

$$L[y] = y'' + y = \cos(t)$$

The function $y = a\cos(t) + b\sin(t)$ will never solve it, because for such a function, $L[y] = 0$. In this case what we must do is multiply our guess by t, or more generally, by a lowest power of t to insure that our guess is not a solution to the homogeneous equation. For this, we must first determine the number of times the root which corresponds to our target solution occurs among the roots of the associated characteristic equation. This can be a trifle tricky to determine; for example, for the equation

$$y'' - 2y' + y = te^t$$

we should guess that our solution is $y = t(at + b)e^t$, since our original guess would be $y = (at + b)e^t$, but this is a solution to the homogeneous equation, while t times it is not; but for

$$y'' - 2y' + y = 3e^t$$

we should guess that our solution is $y = at^2e^t$, since te^t is still a solution to the homogeneous equation, but $y = t^2e^t$ is not.

Finally, if our function $g(t)$ is a linear combination of such functions, we can use this method to solve $L[y] =$ each piece, and then use the Principle of Superposition to find our solution by taking a linear combination.

Higher order equations: The method of undetermined coefficients works equally well for higher order inhomogeneous equations with constant coefficients; the exact same steps will lead you to a solution.

Spring - Mass problems

Basic setup: an object with mass m is attached to a spring along a track. If a force of F_0 is used to pull the mass, stretching the spring, it displaces a distance of u_0. The mass is then displaced from this equilibrium position and released (with some initial velocity). Position at time t is $u(t)$.
Newton: $mu'' =$ sum of the forces acting on the object. These include:
gravity: $F_g = mg$
the spring: $F_s = -ku$ (Hooke's Law)
friction: $F_f = -\gamma u'$
a possible external force: $F_e = f(t)$
Force of F_0 displaces spring by u_0: $F_0 - ku_0 = 0$ (applied force and spring force balance; use to compute k !)
So: $mu'' = -ku - \gamma u' + f(t)$, i.e.,

$$mu'' + ku + \gamma u' = f(t)$$

Some special cases:
No friction ($\gamma = 0$) = undamped, no external force (= free vibration); solutions are

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C \cos(\omega_0 t - \delta)$$

where $\omega = \sqrt{k/m}$ = the *natural frequency* of the system, C = amplitude of the vibration, δ (= 'delay') = phase angle
$C = \sqrt{(c_1^2 + c_2^2)}$, $\tan(\delta) = c_2/c_1$
[[**You are not responsible for these formulas; they are included FYI only.**]]
$T = 2\pi/\omega_0$ = period of the vibration. Note: stiffer spring (= larger k) gives higher frequency, shorter period. Larger m gives the opposite.

Damped free vibrations; solutions depend on $\gamma^2 - 4km$ = discriminant
$\gamma^2 > 4km$ (overdamped); fundamental solutions are $e^{r_1 t}, e^{r_2 t}$, $r_1, r_2 < 0$
$\gamma^2 = 4km$ (critically damped); fundamental solutions are e^{rt}, te^{rt}, $r < 0$
$\gamma^2 < 4km$ (underdamped); fundamental solutions are $e^{rt}\cos(\omega t), e^{rt}\sin(\omega t)$, $r < 0$, $\omega = \sqrt{\omega_0^2 - (\gamma/2m)^2}$
In each case, solutions tend to 0 as t goes to ∞. In first two cases, the solution has at most on local max or min; in the third case, note that the frequency of the periodic part of the motion is smaller than the natural frequency. $T = 2\pi/\omega$ is called the *quasi-period* of the vibration.

Forced vibrations
Focus on periodic forcing term: $f(t) = F_0 \cos(\omega t)$.
Undamped: if $\omega \neq \omega_0$, then (using undetermined coefficients) solution is

$$u = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

This is the sum of two vibrations with different frequencies.
In the special case $u(0) = 0, u'(0) = 0$ (starting at rest), we can further simplify:

$$u = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin(\frac{\omega_0 - \omega}{2}t)\sin(\frac{\omega_0 + \omega}{2}t)$$

When ω is close to ω_0, this illustrates the concept of *beats*; we have a high frequency vibration (the second sine) with *amplitude* a low frequency vibration (the first sine). the mass essentially vibrates rapidly between to sine curves.

When $\omega = \omega_0$, our forcing term is a solution to the homogeneous equation, so the general solution, instead, is

$$u = C\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0}t\sin(\omega t)$$

In this case, as t goes to ∞, the amplitude of the second oscillation goes to ∞; the solution, essentially, resonates with the forcing term. (Basically, you are 'feeding' the system at it's natural frequency.) This illustrates the phenomenon of *resonance*.

Finally, if we include friction ($\gamma \neq 0$), then the solution turns out to be

$$u = \text{homog. soln.} + C\cos(\omega t - \delta), \text{ where}$$

$$C = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \tan(\delta) = \frac{\gamma m}{m(\omega_0^2 - \omega^2)}$$

But since $\gamma > 0$, the homogeneous solutions will tend to 0 as $t \to \infty$; they are called the *transient solution*. (Basically, they just allow us to solve any initial value problem. We can then conclude that any energy given to the susystem is dissipated over time; leaving only the energy imparted by the forcing term to drive the system along.) The other term is called the *forced response*, or *steady-state solution*.
Note that the amplitude C of the forced response goes to 0 as the driving frequency, ω, goes to ∞. Notice also that $\tan(\delta)$ can never be 0, so the forced response is always out of phase with the forcing term. When we are driving the system at it's natural frequency ω_0, the system is 90 degrees out of phase; as $\omega \to \infty$, the system approaches being 180 degrees out of phase, i.e., the motion of the mass is almost exactly <u>opposite</u> to the force being externally applied!

Systems of Equations

Basic idea: we have several unknown functions $x, y, \ldots$ of time t , and equations describing the derivatives of each in terms of $t, x, y, \ldots$. The goal: determine the functions $x(t), y(t), \ldots$ which solve all the equations *at the same time*. Initial value problem: the value of each function is specified at a specific time: $x(t_0) = x_0, y(t_0) = y_0, \ldots$.

Example: Multiple tank problem: several tanks connected by pipes, with solutions of varying concentrations in them. Applying

rate of change of amount of solute =
(rate at which solute comes in) − (rate at which solute goes out)

to each tank gives a system of equations.

A multiple tank problem gives a *linear* system of equations: each equation has the form

$$y_i' = a_1 y_1 + \cdots a_n y_n + f_i(t)$$

for some collection of (often constant) functions of t, $a_1, \ldots, a_n$, and (usually constant) function $f_i(t)$. When the a_i are constant, such systems can be solved by the *elimination method*; the first equation can be rewritten as $y_n =$ (some expression), which can then be substituted into the remaining equations to give $n-1$ equations in $n-1$ unknown functions. The process can then be repeated, yielding, in the end, a single n-th order linear equation (with constant coefficients) in a single unknown function. This can then be solved by our earlier techniques.

[**For further details, see the handout from class!**]

Autonomous systems: The other kind of system of equations which earlier techniques can help us solve is *autonomous systems*; this means that each equation in the system has the form

$$y_i' = f_i(y_1, \ldots, y_n)$$

with no independent variable t appearing on the right-hand side. For these, we can use the *direction field* just as we did for autonomous equations before. For the sake of our exposition, we will deal with an autonomous system of two equations

$$x' = f(x, y) \qquad , \qquad y' = g(x, y)$$

Our solution would be a pair of functions $(x(t), y(t))$, which we can think of as describing a *parametrized curve* in the x-y plane. The tangent vector to this curve is $(x'(t), y'(t))$ $= (f(x(t), y(t)), g(x(t), y(t)) = (f(x, y), g(x, y))$. So every solution curve is tangent to the *direction field* $(f(x, y), g(x, y))$ at every point along the curve. So by drawing the direction field, we can estimate the *trajectories* of solutions (but not, in general, their actual parametrizations), by finding curves tangent to the vectors of the field.

The task of drawing a direction field can be simplified by drawing the *nullclines* of the field, that is, the curves where $f(x, y) = 0$ (vertical tangents) and where $g(x, y) = 0$ (horizontal tangents). Where such curves cross, we have *equilibrium points.* These are points where $x'(t) = y'(t) = 0$; that is, the constant x- and y-values are *constant solutions* to the system of equations.

For a linear system with constant coefficients, the basic shape of the direction field is completely determined by the roots of the auxiliary equation for the second order equation obtained from the elimination method. Complex roots give a spiral pattern around the equilibrium values (and spiralling in or out depending upon whether or not the real part of the roots are negative or not); real roots will make the direction field point towards the equilibrium or away from it, depending on if they are negative or positive; if there is one of each we have one direction pointing in and one pointing out. Details of this may be found on the handout from class!

For a more general autonomous system, we can *linearize* the equation at each equilibrium point (x_0, y_0), that is, replace our original equations with

$$x' = [f_x(x_0, y_0)]x + [f_y(x_0, y_0)]y \qquad , \; y' = [g_x(x_0, y_0)]x + [g_y(x_0, y_0)]y$$

The behavior of solution curves, near the equilibrium point, are well-represented by the solutions to the linearized equation.