Math 106 Exam 2 Topics

The Chain Rule

Composition $(g \circ f)(x_0) = g(f(x_0))$; (note: we <u>don't</u> know what $g(x_0)$ is.)
$(g \circ f)'$ ought to have something to do with $g'(x)$ and $f'(x)$
in particular, $(g \circ f)'(x_0)$ should depend on $f'(x_0)$ and $g'(f(x_0))$

Chain Rule: $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$
$= (d(\text{outside})$ eval'd at inside fcn$)\cdot(d(\text{inside}))$
Ex: $((x^3 + x - 1)^4)' = (4(x^3 + 1 - 1)^3)(3x^2 + 1)$

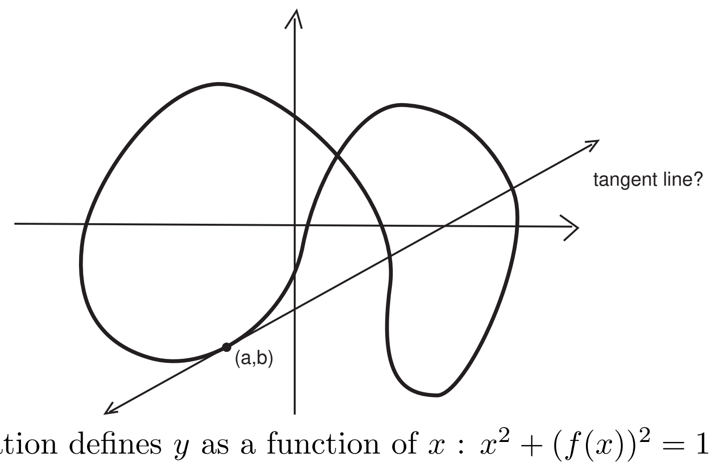
Different notation:
$y = g(f(x)) = g(u)$, where $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Parametric equations: a general curve needn't be the graph of a function. But we can imagine ourselves travelling along a curve, and then $x = x(t)$ and $y = y(t)$ are functions of t=time. We still may have a reasonable tangent line to the graph, and its slope should still be

(change in y/change in x $= \lim\limits_{t\to t_0} \dfrac{y(t) - y(t_0)}{x(t) - x(t_0)} = \lim\limits_{t\to t_0} \dfrac{(y(t) - y(t_0))/(t - t_0)}{(x(t) - x(t_0))/(t - t_0)}$
$= \dfrac{\lim_{t\to t_0}(y(t) - y(t_0))/(t - t_0)}{\lim_{t\to t_0}(x(t) - x(t_0))/(t - t_0)} = \dfrac{y'(t_0)}{x'(t_0)}$

Implicit differentiation

We can differentiate functions; what about *equations*? (e.g., $x^2 + y^2 = 1$)
graph <u>looks</u> <u>like</u> it has tangent lines



Idea: <u>Pretend</u> equation defines y as a function of x : $x^2 + (f(x))^2 = 1$ and differentiate!
$2x + 2f(x)f'(x) = 0$; so $f'(x) = \dfrac{-x}{f(x)} = \dfrac{-x}{y}$

Different notation:
$x^2 + xy^2 - y^3 = 6$; then $2x + (y^2 + x(2y\dfrac{dy}{dx}) - 3y^2\dfrac{dy}{dx} = 0$
$\dfrac{dy}{dx} = \dfrac{-2x - y^2}{2xy - 3y^2}$

Application: extend the power rule

$\frac{d}{dx}(x^r) = rx^{r-1}$ works for any *rational* number r

$(y = x^{p/q}$ means $y^q = x^p$; differentiate!)

Inverse functions and their derivatives

Basic idea: run a function backwards

y=f(x) ; 'assign' the value x to the input y ; x=g(y)

need g a function; so need f is one-to-one

f is one-to-one: if f(x)=f(y) then x=y ; if $x \neq y$ then $f(x) \neq f(y)$

$g = f^{-1}$, then $g(f(x)) = x$ and $f(g(x)) = x$ (i.e., $g \circ f$=Id and $f \circ g$=Id)

finding inverses: rewrite y=f(x) as x=some expression in y

graphs: if (a,b) on graph of f, then (b,a) on graph of f^{-1}

graph of f^{-1} is graph of f, reflected across line y=x

horizontal lines go to vertical lines; horizontal line test for inverse

derivative of the inverse: $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$

Logarithms

$f(x)$=a^x is **either** always increasing $(a > 1)$ **or** always decreasing $(a < 1)$

inverse is $g(x) = \log_a x = \frac{\ln x}{\ln a}$

$\ln x$ is the inverse of e^x.

$\ln x$ is a log; it turns products into sums: $\ln(ab) = \ln(a) + \ln(b)$

$\ln(a^b) = b\ln(a)$; $\ln(a/b) = \ln(a) - \ln(b)$

$e^{\ln x} = x$ and $(e^x)' = e^x$, so $1 = (e^{\ln x})' = (e^{\ln x})(\ln x)' = x(\ln x)'$, so $(\ln x)' = 1/x$.

$\frac{d}{dx}(\ln x) = 1/\text{x}$; $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ This gives us:

Logarithmic differentiation: $f'(x) = f(x)\frac{d}{dx}(\ln(f(x)))$

useful for taking the derivative of products, powers, and quotients

$\ln(a^b)$ **should be** $b\ln a$, so $a^b = e^{b\ln a}$

$a^{b+c} = a^b a^c$; $a^{bc} = (a^b)^c$

$a^x = e^{x\ln a}$; $\frac{d}{dx}(a^x) = a^x \ln a$

$x^r = e^{r\ln x}$ (makes sense for *any real number* r) ; $\frac{d}{dx}(x^r) = e^{r\ln x}(r)(\frac{1}{x}) = rx^{r-1}$

Inverse trigonometric functions

Trig functions ($\sin x$, $\cos x$, $\tan x$, etc.) aren't one-to-one; make them!

$\sin x$, $-\pi/2 \leq x \leq \pi/2$ is one-to-one; inverse is Arcsin x

sin(Arcsin x)=x, all x; Arcsin($\sin x$)=x IF x in range above

$\tan x$, $-\pi/2 < x < \pi/2$ is one-to-one; inverse is Arctan x

tan(Arctan x)=x, all x; Arctan($\tan x$)=x IF x in range above

$\sec x$, $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$, is one-to-one; inverse is Arcsec x
sec(Arcsec$x$)=x, all x; Arcsec($\sec x$)=x IF x in range above

Computing cos(Arcsin x), tan(Arcsec x), etc.; use right triangles

The other inverse trig functions aren't very useful,
they are essentially the negatives of the functions above.

Derivatives of inverse trig functions

They are the derivatives of inverse functions! Use right triangles to simplify.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{x^2+1}$$
$$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{\sec(\text{arcsec } x)\tan(\text{arcsec } x)} = \frac{1}{|x|\sqrt{x^2-1}}$$

Related Rates

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$; pretend each is a function of t, and differentiate (implicitly).

General procedure:
Draw a picture, describing the situation; label things with variables.
Which variables, rates of change do you know, or want to know?
Find an equation relating the variables whose *rates of change* you know or want to know.
Differentiate!
Plug in the values that you know.

Linear approximation and differentials

Idea: The tangent line to a graph of a function makes a good approximation to the function, near the point of tangency.
Tangent line to $y = f(x)$ at $(x_0, f(x_0) : L(x) = f(x_0) + f'(x_0)(x - x_0)$
$f(x) \approx L(x)$ for x near x_0
Ex.: $\sqrt{27} \approx 5 + \frac{1}{2\cdot 5}(27-25)$, using $f(x) = \sqrt{x}$
$(1+x)^k \approx 1 + kx$, using x_0=0
$\Delta f = f(x_0 + \Delta x) - f(x_0)$, then $f(x_0 + \Delta x) \approx L(x_0 + \Delta x)$ translates to
$\Delta f \approx f'(x_0) \cdot \Delta x$
differential notation: $df = f'(x_0)dx$
So $\Delta f \approx df$, when $\delta x = dx$ is small
In fact, $\Delta f - df$ = (diffrnce quot $-f'(x_0))\Delta x$ = (small)$\cdot$(small) = really small, goes like $(\Delta x)^2$

Chapter 4: Applications of Derivatives

Extreme Values

c is an (absolute) maximum for a function $f(x)$ if $f(c) \geq f(x)$ for every other x
d is an (absolute) minimum for a function $f(x)$ if $f(d) \leq f(x)$ for every other x
max or min = extremum

Extreme Value Theorem: If f is a continuous function defined on a closed interval $[a, b]$, then f actually *has* a max and a min.

Goal: figure out where they *are*!
c is a relative max (or min) if $f(c)$ is $\geq f(x)$ (or $\leq f(x)$) for every x *near* c. Rel max or min = rel extremum.
An absolute extremum is either a rel extremum or an endpoint of the interval.

c is a critical point if $f'(c) = 0$ or does not exist.
A rel extremum is a critical point.

So absolute extrema occur either at critical points *or* at the endpoints.
So to find the abs max or min of a function f on an interval $[a, b]$:
(1) Take derivative, find the critical points.
(2) Evaluate f at each critical point and endpoint.
(3) Biggest value is maximum value, smallest is minimum value.

The Mean Value Theorem

You can (almost) recreate a function by knowing its derivative

Mean Value Theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences:
Rolle's Theorem: $f(a) = f(b) = 0$; between two roots there is a critical point.
So: If a function has no critical points, it has at *most* one root!
A function with $f'(x)$=0 is constant.
Functions with the same derivative (on an interval) differ by a constant.

The First Derivative Test

f is *increasing* on an interval if $x > y$ implies $f(x) > f(y)$
f is *decreasing* on an interval if $x > y$ implies $f(x) < f(y)$
If $f'(x) > 0$ on an interval, then f is increasing
If $f'(x) < 0$ on an interval, then f is decreasing

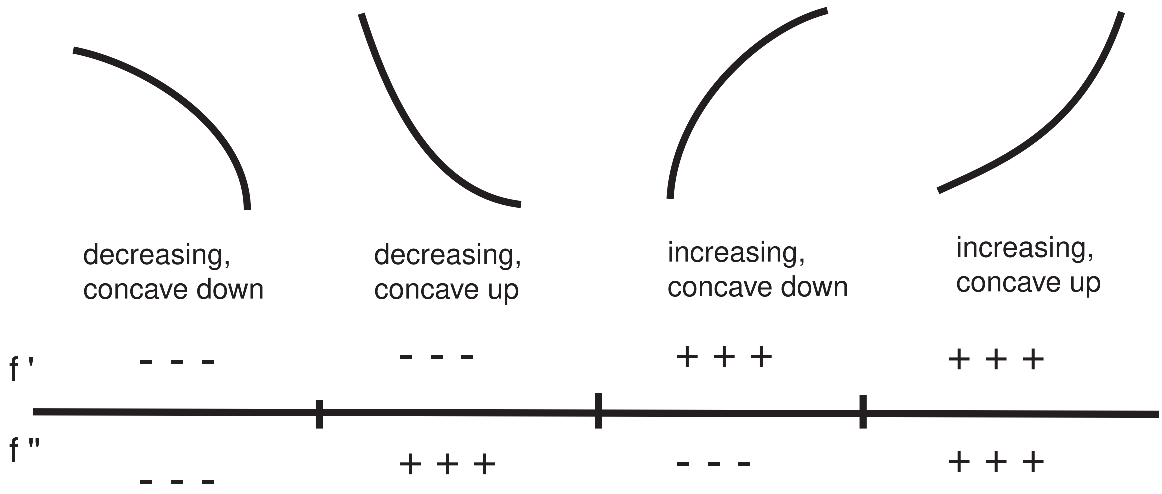
Local max's / min's occur at critical points; how do you tell them apart?
Near a local max, f is increasing, then decreasing; $f'(x) > 0$ to the left of the critical point, and $f'(x) < 0$ to the right.
Near a local min, the opposite is true; $f'(x) < 0$ to the left of the critical point, and $f'(x) > 0$ to the right.
If the derivative does *not* change sign as you cross a critical point, then the critical point is not a rel extremum.
Basic use: plot where a function is increasing/decreasing: plot critical points; in between them, sign of derivative does not change.

The second derivative test and graphing

When we look at a graph, we see where function is increasing/decreasing. We also see:

f is concave up on an interval if $f''(x) > 0$ on the interval
Means: f' is increasing; f is *bending* up.
f is concave down on an interval if $f''(x) < 0$ on the interval
Means: f' is decreasing; f is *bending* down.
A point where the concavity changes is called a point of inflection

Graphing:
Find where $f'(x)$ and $f''(x)$ are 0 or DNE
Plot on the same line.
In between points, derivative and second derivative don't change sign, so graph looks like one of:



Then string together the pieces!
Use information about vertical and horizontal asymptotes to finish sketching the graph.

Second derivative test: If c is a critical point and
$f''(c) > 0$, then c is a rel min (smiling!)
$f''(c) < 0$, then c is a rel max (frowning!)