# Math 208H, Section 1

## Practice problems for Exam 2 (Solutions)

[**Disclaimer:** these solutions were written somewhat hastily and without much verification, so while the method described is almost certainly correct, the actual computations do not carry the same claims of correctness....]

**A1.** Find the local extrema of the function $f(x,y) = 2x^4 - 2xy + y^2$ , and determine, for each, if it is a local max. local min, or saddle point.

Local extrema occur at critical points, so we compute: $f_x = 8x^3 - 2y$ and $f_y = -2x + 2y$ . These are never undefined, so our only critical points will occur when both are 0. $f_y = -2x + 2y = 0$ means $2y = 2x$, so $y = x$. Substituting this into $f_x = 8x^3 - 2y = 0$ gives $8x^3 - 2x = (2x)(4x^2 - 1) = 0$, so either $x = 0$, or $4x^2 - 1 = 0$, so $x = 0$ or $x = 1/2$ or $x = -1/2$. This yields the three critical points $(0,0)$ , $(1/2, 1/2)$ , and $(-1/2, -1/2)$.

To determine their character, we need the Hessian: $f_{xx} = 24x^2$, $f_{xy} = -2$, and $f_{yy} = 2$, so $H = f_{xx}f_{yy} - (f_{xy})^2 = 48x^2 - 4$. At $(0,0)$ $H = -4 < 0$, so $(0,0)$ is a saddle point. At $(1/2, 1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$, so $(1/2, 1/2)$ is a local min. And at $(-1/2, -1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$ as well, so $(-1/2, -1/2)$ is also a local min.

**A2.** Find the point(s) on the ellipse $g(x,y) = x^2 + 3y^2 = 4$ where the function $f(x,y) = x - 3y + 4$ achieves it maximum value.

We use Lagrange multipliers, which requires us to solve

$1 = \lambda(2x)$ , $-3 = \lambda(6y)$, and $x^2 + 3y^2 = 4$

The first two equations tell us that $\lambda$ cannot be 0, and so we can solve them for $x$ and $y$ and plug into the third equation, which yields (after clearing denomenators) $4 \cdot 4\lambda^2 = 4$, so $\lambda = \pm 1/2$. Using these values in our first two equations yields $(x,y) = (1,-1)$ or $(-1,1)$. Plugging into $f$, we find that the maximum occurs at $(1,-1)$.

**A3.** Evaluate the iterated integral $\displaystyle\int_0^2 \int_x^2 x^2(y^4+1)^{1/3}\, dy\, dx$

by rewriting the integral to reverse the order of integration. (Note: the integral *cannot* be evaluated in the order given....)

The region $x \le y \le 2$, for $0 \le x \le 2$, is a triangle formed by the lines $y = x$, $y = 2$, and $x = 0$. Writing this as a collection of horizontal lines gives the alternate decription $0 \le x \le y$ for $0 \le y \le 2$. This yields the alternate iterated integral $\displaystyle\int_0^2 \int_0^y x^2(y^4+1)^{1/3}\, dx\, dy = \int_0^2 \frac{y^3}{3}(y^4+1)^{1/3}\, dy = \frac{3}{4}\frac{1}{12}(y^4+1)^{4/3}\Big|_0^2 = \frac{1}{16}[(2^4+1)^{4/3} - 1]$ .

**A4.** Find the integral of the function $f(x,y,z) = x + y + z$ over the region lying between the graph of $z = x^2 + y^2 - 4$ and the $x$-$y$ plane.

The graph of $z$ is a paraboloid, lowered by 4 units, and so the region betweeen lies above the paraboloid and below the plane. The vertical lines which hit the region are those with $x^2 + y^2 \le 4$, which described the inside of the circle of radius 2 centered at the origin. So this integral is perhaps best set up using cylindrical coordinates: The shadow $R$ is given by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$ in polar coordinates. So the integral is:

$$\int_R \int_{x^2+y^2-4}^{0} x + y + z \; dz \; dA = \int_0^{2\pi} \int_0^2 \int_{r^2-4}^{0} r\cos\theta + r\sin\theta + z \; r \; dz \; dr \; d\theta \; .$$

We omit the iterated integral calculation; you should carry it through!

**A5.** Find the integral of the function $f(x,y) = xy^2$ over the region lying in the first quadrant of the $x$-$y$ plane and lying inside of the circle $x^2 + y^2 = 9$ .

The region $R$ is 'best' described in polar coordinates, as $0 \le r \le 3$ and $0 \le \theta \le 2\pi$. The Jacobian for this change of variables is $r$, and $f(x,y) = xy^2 = r^3 \cos\theta \sin^2\theta$, yielding the integral $\int_0^{2\pi} \int_0^3 r^4 \cos\theta \sin^2\theta \; dr \; d\theta = \int_0^{2\pi} \frac{32}{5} \cos\theta \sin^2\theta \; d\theta = \frac{32}{5} \frac{\sin^3\theta}{3} \Big|_0^{2\pi} = 0 - 0 = 0$

**A6.** Find the integral of the function $f(x,y) = 6x + y^2$ over the region in the $x$-$y$ plane between the $x$-axis and the lines $y = x$ and $y = 6 - 2x$ .

We can set this up several ways. The region is a triangle resting on the $x$-axis. Integrating $dy$ first would require us to dut it into two pieces, so let's try the other way. The region is $y \le x \le (6-y)/2$ for $0 \le y \le$ the point where the two lines meet; $x = 6 - 2x$ when $x = 2$, so $0 \le y \le 2$. The integral then becomes $\int_0^2 \int_y^{(6-y)/2} 6x + y^2 \; dx \; dy = \int_0^2 3x^2 + y^2 x \Big|_y^{(6-y)/2} dy = \int_0^2 3over4(6-y)^2 + \frac{1}{2} y^2 (6 - y) - 3y^2 - y^3 \; dy$, which finishes as a slightly ugly but otherwise routine integral.

**A7.** Find the integral of the function $f(x,y) = xy^2$ over the region in the plane lying between the graphs of $a(x) = 2x$ and $b(x) = 3 - x^2$ .

Our first task is to describe the region. To do this we need to know where the graphs meet, so we solve $2x = 3 - x^2$, so $0 = x^2 + 2x - 3 = (x+3)(x-1)$, so $x = -3, 1$. Between these two points we have $(x+3)(x-1) < 0$, so $2x < 3 - x^2$. So the region is $2x \le y \le 3 - x^2$, for $-3 \le x \le 1$. So our integral is $\int_{-3}^{1} \int_{2x}^{3-x^2} xy^2 \; dy \; dx$. This equals $\int_{-3}^{1} \frac{xy^3}{3} \Big|_{2x}^{3-x^2} dx = \frac{1}{3} \int_{-3}^{1} x(3-x^2)^3 - 8x^4 \; dx$ . Noting that the first piece of the integrand is nicely arranged for a $u$-substitution ($u = 3 - x^2$) can make the rest of the computation a bit more pleasant...

**A8.** Evaluate the following double integrals:

(a): $\int_0^1 \int_1^2 x^2 y - y^2 x \; dx \; dy$

Having no particular reason to switch the order of integration, we find that the integral equals $\int_0^1 \frac{1}{3}x^3 y - \frac{1}{2} y^2 x^2 \Big|_{x=1}^{x=2} dy = \int_0^1 (\frac{1}{3} 8y - \frac{1}{2} 4y^2) - (\frac{1}{3} y - \frac{1}{2} y^2) \; dy = \int_0^1 \frac{7}{3} y - \frac{3}{2} y^2 \; dy = \frac{7}{6} y^3 - \frac{1}{2} y^3 \Big|_0^1 = \frac{7}{6} - \frac{1}{2} = \frac{7-3}{6} = \frac{2}{3}$ [although don't count on that...]

(b): $\int_0^1 \int_{\sqrt{x}}^1 x\sqrt{y} \; dy \; dx$

We can go at this straight ahead, as written, or, for fun, switch the order of integration, since $y = \sqrt{x}$ and $y = 1$ meet at $x = 1$, which is the other limit of integration. Drawing a figure, we find that the region has the alternate description $0 \le x \le y^2$ for $0 \le y \le 1$, so the integral equals

$$\int_0^1 \int_0^{y^2} x\sqrt{y}\ dx\ dy = \int_0^1 \frac{x^2\sqrt{y}}{2}\Big|_{x=0}^{x=y^2}\ dy = \int_0^1 \frac{1}{2}x^{9/2} = \frac{2}{11}\frac{1}{2}x^{11/2}\Big|_0^1 = \frac{1}{11} - 0 = \frac{1}{11}\ .$$

**A9.** Find the integral of the function $f(x, y) = x$ over the region $R$ lying between the graphs of the curves

$$y = x - x^2 \text{ and } y = x - 1.$$

This is much like a previous problem; The two graphs meet when $x - x^2 = x - 1$, so $x = -1, 1$ and between these numbers $x - 1 \le x - x^2$, so our region is $x - 1 \le y \le x - x^2$ for $-1 \le x \le 1$. So our integral is

$$\int_{-1}^1 \int_{x-1}^{x-x^2} x\ dy\ dx = \int_{-1}^1 xy\Big|_{y=x-1}^{y=x-x^2}\ dx = \int_{-1}^1 (x^2-x^3)-(x^2-x)\ dx = \int_{-1}^1 x-x^3\ dx =$$

$x^2/2 - x^4/4\Big|_{-1}^1 = (1/2 - 1/4) - (1/2 - 1/4) = 0$ . [Hm, that seems to happen a lot...]

**A10.** Use Lagrange multipliers to find the maximum value of the function $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + 4y^2 - 1 = 0$ .

Setting the gradients equal (with multiplier $\lambda$), we wish to solve $y = \lambda(2x)$ , $x = \lambda(8y)$ , and $x^2 + 4y^2 = 1$ . This means $y = 2\lambda x = 2\lambda(8\lambda y) = 16\lambda^2 y$, so either $y = 0$ or $16\lambda^2 = 1$, so $\lambda = \pm 1/4$. But if $y = 0$ then $x = 8\lambda y = 0$, which will not satisfy $x^2 + 4y^2 = 1$, so that won't work.

So $\lambda = \pm 1/4$, giving us $x = \pm 2y$, so $(\pm 2y)^2 + 4y^2 = 8y^2 = 1$, so $y = \pm\sqrt{2}/4$. This gives us four points:

$(x, y) = (-\sqrt{2}/2, -\sqrt{2}/4), (-\sqrt{2}/2, \sqrt{2}/4)4, (\sqrt{2}/2, -\sqrt{2}/4)4$, or $(\sqrt{2}/2, \sqrt{2}/4)4$.

Plugging into $f$ gives two values; the larger is $1/4$ [and the smaller is $-1/4$].

**A11.** Find the area of the region $S$ bounded by one loop of the curve described by

$r = \sin(3\theta)$

in polar coordinates. (Hint; to determine the limits of integration, when is $r = 0$?)

The first return to $r = 0$ after $\theta = 0$ is when $3\theta = \pi$, so $\theta = \pi/3$. This gives us the integral $\displaystyle\int\int_S dx\ dy = \int_0^{\pi/3}\int_0^{\sin(3\theta)} r\ dr\ d\theta = \frac{1}{2}\int_0^{\pi/3} r^2\Big|_0^{\sin(3\theta)}\ d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\ d\theta =$

$$\frac{1}{4}\int_0^{\pi/3}(1 - \cos(6\theta)\ d\theta = \frac{1}{4}(\theta - \frac{1}{6}\sin(6\theta))\Big|_0^{\pi/3} = \frac{\pi}{12}$$

**A12.** A particle is moving through 3-space along the parametrized curve $\vec{r}(t) = (\cos t, \sin t, t^{3/2})$ . Find:

(a) the velocity of the particle at time $t$,

Taking derivatives, $\vec{r}'(t) = (-\sin t, \cos t, (3/2)t^{1/2}$.

(b) the acceleration of the particle at time $t$, and

Taking derivatives again, $\vec{r}''(t) = (-\cos t, -\sin t, (3/4)t^{-1/2})$.

(c) the length of the curve traced out by the particle between $t = 0$ and $t = 2$ .

We need to compute the speed: $||\vec{r}'(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2 + ((3/2)t^{1/2})^2}$
$= \sqrt{\sin^2 t + \cos^2 t + (9/4)t} = \sqrt{(9/4)t + 1} = (3/2)\sqrt{t + (4/9)}.$

To find the length of the curve, we integrate:

Length $= \int_0^2 (3/2)\sqrt{t + (4/9)}\ dt = (3/2) \int_{4/9}^{22/9} \sqrt{u}\ du$

$= (3/2)(2/3)u^{3/2}\Big|_{4/9}^{22/9} = (22/9)^{3/2} - (4/9)^{3/2}$