Math 208H, Section 1

Practice problems for Exam 1 (Solutions)

1. Find the **sine** of the angle between the vectors (1,–1,2) and (1,2,1) .

We can use the dot product (dividing by lengths) to compute the cosine of the angle, and then from that the sine. Or we can use $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$ to compute the sine, by finding the cross product and computing lengths.

$\sin(\theta) = \sqrt{(-5)^2 + 1^2 + 3^2}/(\sqrt{1^2 + (-1)^2 + 2^2} \cdot \sqrt{1^2 + 2^2 + 1^2}) = \sqrt{35}/(\sqrt{6} \cdot \sqrt{6}) = \sqrt{35}/6$

This is consistent with $\cos(\theta) = (1 \cdot 1 + (-1) \cdot 2 + 2 \cdot 1)/(\sqrt{6} \cdot \sqrt{6}) = 1/6$.

2. Find a vector of length 3 that is perpendicular to both

$$\vec{v} = \langle 1, 3, 5 \rangle \text{ and } \vec{w} = \langle 2, 1, -1 \rangle .$$

A vector perpendicular to both is given by the cross product, so we compute

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= \langle -3 - 5, -(-1 - 10), 1 - 6 \rangle = \langle -8, 11, -5 \rangle$$

[We can test that this is perpendicular to the two vectors by computing dot products...]

This vector has length $\sqrt{64 + 121 + 25} = \sqrt{210}$; since we want a vector of length 3, we take the appropriate scalar multiple:

$\vec{N} = \dfrac{3}{\sqrt{210}}\langle -8, 11, -5 \rangle$ has length 3 and is $\perp$ tp $\vec{v}$ and $\vec{w}$. [Its negative also works...]

3. Show that if the vectors $\vec{\mathbf{v}}$=(a_1, a_2, a_3) and $\vec{\mathbf{w}}$=(b_1, b_2, b_3) have the same length, then the vectors $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ and $\vec{\mathbf{v}} - \vec{\mathbf{w}}$ are perpendicular to one another.

We wish to know that $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \circ (\vec{\mathbf{v}} - \vec{\mathbf{w}}) = 0$. But expanding this out, we find that it is equal to $\vec{\mathbf{v}} \circ \vec{\mathbf{v}} - \vec{\mathbf{w}} \circ \vec{\mathbf{w}}$. This will be equal to 0 precisely when $|\vec{v}|^2 = \vec{\mathbf{v}} \circ \vec{\mathbf{v}} = \vec{\mathbf{w}} \circ \vec{\mathbf{w}} = |\vec{w}|^2$. This in turn, means that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same length.

4. Find the equation of the plane in 3-space which passes through the three points $(1, 2, 1)$, $(6, 1, 2)$, and $(9, -2, 1)$. Does the point $(3, 2, 1)$ lie on this plane?

To find the equation, we need a point and a normal vector; the normal can be found by a cross product. $\vec{N} = \vec{PQ} \times \vec{PR} = (5, -1, 1) \times (8, -4, 0) = (4, 8, -12)$. Then the equation is $(4, 8, -12) \circ (x - 1, y - 2, z - 1) = 0$, or $4x + 8y - 12z = 8$ (or $x + 2y - 3z = 2$ (!)). [Check: the 3 points satisfy the equation!] Checking, $4 \cdot 3 + 8 \cdot 2 - 12 \cdot 1 = 16 \neq 8$, so the point does not lie on the plane.

5. Find the partial derivatives of the following functions:

(a) $f(x, y, z) = x\tan(2x + yz)$

We have $f_x = \tan(2x + yz) + x\sec^2(2x + yz) \cdot 2$, $f_y = x\sec^2(2x + yz) \cdot z$, and $f_z = x\sec^2(2x + yz) \cdot y$.

(b) $g(x, y) = \dfrac{x^2y - ty^4}{\sin(3y) + 4}$ We have $g_x = \dfrac{(2xy)(\sin(3y) + 4) - (x^2y - ty^4)(0)}{(\sin(3y) + 4)^2}$, and $g_y = \dfrac{(x^2-4ty^3)(\sin(3y)+4)-(x^2y-ty^4)(3\cos(3y))}{(\sin(3y)+4)^2}$. Since the question didn't ask us to do anything with these, why simplify them?

6. Find the equation of the tangent plane to the graph of the equation $f(x,y,z) = xy^2+x^2z-xyz = 5$, at the point $(-1,1,3)$.

$f_x = y^2+2xz-yz$, $f_y = 2xy+x^2-xz$, and $f_z = x^2-xy$. The normal vector to the plane will be $(f_x(-1,1,3), f_y(-1,1,3), f_z(-1,1,3)) = (1-6-3,-2+1+3,1+1) = (-8,2,2)$. Together with the point of tangency, this gives us the equation

$-8(x-(-1))+2(y-1)+2(z-3)=0$, or $-8x+2y+2z=16$, or $4x-y-z=-8$.

7. Calculate the first and second partial derivatives of the function $h(x,y) = \dfrac{\sin(x+y)}{y}$

It may help a bit to write this function as $h(x,y) = y^{-1}\sin(x+y)$. Then we have

$h_x = y^{-1}\cos(x+y)\cdot 1 = y^{-1}\cos(x+y)$,

$h_y = -y^{-2}\sin(x+y)+y^{-1}\cos(x+y)\cdot 1 = -y^{-2}\sin(x+y)+y^{-1}\cos(x+y)$. Then

$h_{xx} = (h_x)_x = y^{-1}(-\sin(x+y)\cdot 1) = -y^{-1}\sin(x+y)$

$h_{yx} = h_{xy} = (h_x)_y = -y^{-2}\cos(x+y)+y^{-1}(-\sin(x+y)\cdot 1)$

$= -y^{-2}\cos(x+y)-y^{-1}\sin(x+y)$

$h_{yy} = (h_y)_y$

$= [2y^{-3}\sin(x+y)-y^{-2}(\cos(x+y)\cdot 1)]+[-y^{-2}\cos(x+y)+y^{-1}(-\sin(x+y)\cdot 1)]$

Again, we don't want to do anything with it, so why bother simplifying it...

8. In which direction is the function $f(x,y) = x^4y-3x^2y^2$ increasing the fastest, at the point (1,2) ? In which directions is the function *neither* increasing *nor* decreasing?

f increases fastest in the direction of the gradient, so we compute:

$\nabla f = (4x^3y-6xy^2, x^4-6x^2y)$, which at (1,2) gives $\vec{v} = (8-24, 1-12) = (-16,-11)$. This is the drection of fastest increase (you can divide by its length if you want a unit vector...).

For no increase/decrease, what we want is $D_{\vec{w}}f = \nabla f \circ \vec{w} = 0$, so we want

$(-16,11)\circ(\alpha,\beta) = -16\alpha-11\beta = 0$; we can do this, for example, with $\vec{w} = (\alpha,\beta) = (11,-16)$. [There are many other answers, all scalar multiples of this one.]

9. If $f(x,y) = x^2y^5-x+3y-4$, $x = x(u,v) = \dfrac{u}{u+v}$ and $y = y(u,v) = uv-u$, use the Chain Rule to find $\dfrac{\partial f}{\partial u}$ when $u=1$ and v=0 .

First, when $(u,v)=(1,0)$, then $x = 1/(1+0) = 1$ and $y = 1\cdot 0-1 = -1$. From the chain rule, we know that $f_u = f_x x_u + f_y y_u$, evaluated at $(x,y)=(1,-1)$ and $(u,v)=(1,0)$. We compute:

$f_x = 2xy^5-1 = -2-1 = -3$, $f_y = 5x^2y^4+3 = 5+3 = 8$, $x_u = \dfrac{(1)(u+v)-(u)(1)}{(u+v)^2} = \dfrac{v}{(u+v)^2} = 0$, and $y_u = v-1 = 0-1 = -1$; so at $(u,v)=(1,0)$ we have $f_u(1,0) = (-3)(0)+(8)(-1) = -8$.

10. If $f(x,y) = \dfrac{x^2y}{x+y}$, and $\gamma(t) = (x(t), y(t))$ is a parametrized curve in the domain of f with $\gamma(0) = (2,-1)$ and $\gamma'(0) = (3,5)$, then what is $\left.\dfrac{d}{dt} f(\gamma(t))\right|_{t=0}$?

By the chain rule, $\dfrac{df}{dt} = f_x x_t + f_y y_t$. We compute: $f_x = \dfrac{(2xy)(x+y) - (x^2y)(1)}{(x+y)^2}$ and $f_y = \dfrac{(x^2)(x+y) - (x^2y)(1)}{(x+y)^2}$.

At $(2,-1)$, these are $f_x = \dfrac{(-4)(1) - (-4)(1)}{(1)^2} = 0$ and $f_y = \dfrac{(4)(1) - (-4)(1)}{(1)^2} = 8$, so $\dfrac{df}{dt} = f_x x_t + f_y y_t = (0)(3) + (8)(5) = 40$.

11. Find the **second** partial derivatives of the function $h(x,y) = x\sin(xy^2)$.

We compute: $h_x = (1)(\sin(xy^2)) + (x)(\cos(xy^2))(y^2) = \sin(xy^2) + xy^2\cos(xy^2)$
$h_y = x(\cos(xy^2))(2xy) = 2x^2y\cos(xy^2)$. Then for the second partials:
$h_{xx} - (h_x)_x = (\cos(xy^2))(y^2) + [(y^2)(\cos(xy^2)) + (xy^2)(-\sin(xy^2))(y^2)]$
$= 2y^2\cos(xy^2) - xy^4\sin(xy^2)$
$h_{xy} = h_{yx} = (h_y)_x = (4xy)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(y^2)$
$= 4xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$
$h_{yy} = (h_y)_y = (2x^2)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(2xy)$
$= 2x^2\cos(xy^2) - 4x^3y^2\sin(xy^2)$

12. For which value(s) of c are the vectors $\vec{v} = (1,2,c)$ and $\vec{w} = (-5,2c,4)$ orthogonal?

We want $\vec{v} \bullet \vec{w} = 0$, so $0 = (1,2,c) \bullet (-5,2c,4) = -5 + 4c + 4c = -5 + 8c$, so $8c = 5$ and so $c = 5/8$. This gives the vectors

$$\vec{v} = (1, 2, \frac{5}{8}) \text{ and } \vec{w} = (-5, \frac{5}{4}, 4) .$$

[As a check, $\vec{v} \bullet \vec{w} = (1,2,\frac{5}{8}) \bullet (-5,\frac{5}{4},4) = -5 + \frac{5}{2} + \frac{5}{2} = 0$, as desired.]

13. Find the equation of the plane passing through the points

$$(2,3,5) \ , \ (1,-1,0) \ , \text{ and } (1,1,2) .$$

Labeling the points P, Q, and R for convenience, we have $\vec{v} = \vec{PQ} = (-1,-4,-5)$ and $\vec{w} = \vec{PR} = (-1,-2,-3)$. These are directions in the plane, and so their cross product will be normal to the plane. So we compute

$$\vec{n} = \vec{v}\times\vec{w} = (\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_1 \end{vmatrix}) = (\begin{vmatrix} -4 & -5 \\ -2 & -3 \end{vmatrix}, -\begin{vmatrix} -1 & -5 \\ -1 & -3 \end{vmatrix}, \begin{vmatrix} -1 & -4 \\ -1 & -2 \end{vmatrix})$$
$= (12 - 10, -(3-5), 2-4) = (2,2,-2)$.

[As a check, we can compute $\vec{v} \bullet \vec{n} = -2 - 8 + 10 = 0$ and $\vec{w} \bullet \vec{n} = -2 - 4 + 6 = 0$.]

With a normal $\vec{n} = (2,2,-2)$ to the plane and a point $P = (2,3,5)$ on the plane, we can give the equation for the plane as $\vec{n} \bullet [(x,y,z) - (2,3,5)] = 0$, i.e.,

$$2(x-2)+2(y-3)-2(z-5)=0\ .$$

[There are, of course, many other equivalent answers, obtained by choosing, for example, another point to use as the tails of our vectors....]

14. What is the rate of change of the function $f(x,y)=\dfrac{xy}{x+2y}$, at the point $(4,2)$, and in the direction of the vector $\vec{v}=(1,1)$?

The rate of change is the directional derivative, computed as $\nabla f(4,2)\bullet\vec{v}$. So we compute:

$f(x,y)=xy(x+2y)^{-1}$, so $f_x=(y)(x+2y)^{-1}+(xy)[(-1)(x+2y)^{-2}(1)]$, and
$f_y=(x)(x+2y)^{-1}+(xy)[(-1)(x+2y)^{-2}(2)]$. So

$\nabla f(4,2)=((2)(8)^{-1}+(8)(-1)(8)^{-2},(4)(8)^{-1}+(8)(-1)(8)^{-2}(2))=(\frac{1}{4}-\frac{1}{8},\frac{1}{2}-\frac{1}{4})=(\frac{1}{8},\frac{1}{4})$

So the rate of change is $\nabla f(4,2)\bullet\vec{v}=(\frac{1}{8},\frac{1}{4})\bullet(1,1)=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$.

[Under some interpretations, we should divide this number by $||(1,1)||=\sqrt{2}$, in order to be using the unit vector pointing in the direction of $\vec{v}$.]

15. Find the equation of the plane tangent to the graph of the function
$g(x,y)=x^3y-4x^2y^2+2xy^4$ at the point $(2,1,g(2,1))$.

We can describe this plane using a point on the plane and its $x-$ and $y-$slopes, all of which the function can provide.

The point of tangency $(2,1,g(2,1))=(2,1,8-16+4)=(2,1,-4)$ is a point on the plane.

For the slopes, we compute:

$g_x=3x^2y-8xy^2+2y^4$, so $f_x(2,1)=12-16+2=-2=m=x-$slope.

$g_y=x^3-8x^2y+8xy^3$, so $f_y(2,1)=8-32+16=-8=n=y-$slope.

So the equation for the tangent plane is given by

$$z=g(2,1)+g_x(2,1)(x-2)+g_y(2,1)(y-1)=-4-2(x-2)-8(y-1)$$

Multiplying out, this can be converted to $z=-4-2x+4-8y+8=-2x-8y+8$.

16. If $x=u^2v$ and $y=uv^2$, then show how to express the partial derivatives of $g(u,v)=f(x(u,v),y(u,v))$ at the point $(u,v)=(2,-1)$, in terms of the (at the moment unknown) partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Writing $z=g(u,v)=f(x(u,v),y(u,v))$, by the Chain rule, we know that $z_u=z_xx_u+z_yy_u$ and $z_v=z_xx_v+z_yy_v$. We can compute

$x(2,-1)=2^2(-1)=-4$ and $y(2,-1)=2(-1)^2=2$, so $(x,y)=(-4,2)$, while

$x_u=2uv$, $x_v=u^2$, $y_u=v^2$, and $y_v=2uv$, and so at $(u,v)=(2,-1)$, we have

$x_u = -4$, $x_v = 4$, $y_u = 1$, and $y_v = -4$. So at $(u, v) = (2, -1)$, we have

$g_u(2, -1) = [f_x(-4, 2)][-4] + [f_y(-4, 2)][1] = -4f_x(-4, 2) + f_y(-4, 2)$, and
$g_v(2, -1) = [f_x(-4, 2)][4] + [f_y(-4, 2)][-4] = 4f_x(-4, 2) - 4f_y(-4, 2)$.

17. Find the **second** partial derivatives of the function $h(x, y) = xe^{xy}$.

We compute:
$h_x = (1)(e^{xy}) + (x)(e^{xy}y) = e^{xy} + xye^{xy}$ and $h_y = (0)(e^{xy}) + (x)(e^{xy}x) = x^2e^{xy}$. So

$h_{xx} = (h_x)_x = (e^{xy})(y) + [(y)e^{xy} + (xy)(e^{xy}y)] = 2ye^{xy} + xy^2e^{xy}$,
$h_{xy} = (h_x)_y = (e^{xy})(x) + [(x)(e^{xy}) + (xy)(e^{xy}x)] = 2xe^{xy} + x^2ye^{xy}$,
$h_{yx} = (h_y)_x = (2x)(e^{xy}) + (x^2)(e^{xy}y) = 2xe^{xy} + x^2ye^{xy} = h_{xy}$, and
$h_{yy} = (x^2)(e^{xy}x) = x^3e^{xy}$.

18. Find the local extrema of the function $f(x, y) = 2x^4 - 2xy + y^2$, and determine, for each, if it is a local max. local min, or saddle point.

Local extrema occur at critical points, so we compute: $f_x = 8x^3 - 2y$ and $f_y = -2x + 2y$. These are never undefined, so our only critical points will occur when both are 0. $f_y = -2x + 2y = 0$ means $2y = 2x$, so $y = x$. Substituting this into $f_x = 8x^3 - 2y = 0$ gives $8x^3 - 2x = (2x)(4x^2 - 1) = 0$, so either $x = 0$, or $4x^2 - 1 = 0$, so $x = 0$ or $x = 1/2$ or $x = -1/2$. This yields the three critical points $(0, 0)$, $(1/2, 1/2)$, and $(-1/2, -1/2)$.

To determine their character, we need the Hessian: $f_{xx} = 24x^2$, $f_{xy} = -2$, and $f_{yy} = 2$, so $H = f_{xx}f_{yy} - (f_{xy})^2 = 48x^2 - 4$. At $(0, 0)$ $H = -4 < 0$, so $(0, 0)$ is a saddle point. At $(1/2, 1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$, so $(1/2, 1/2)$ is a local min. And at $(-1/2, -1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$ as well, so $(-1/2, -1/2)$ is also a local min.