Math 208H, Section 1

Quiz number 5 Solutions

For the function

$$f(x,y) = 5x^2 + x^2y - 12x - \frac{1}{3}y^3$$

find the critical points of f, and <u>for one of them</u> (your choice), determine if the critical point is a relative maximum, relative minimum, or saddle point for the function f.

To find the critical points, we compute:

$$f_x = 10x + 2xy - 12 = 0 \qquad \text{and} \qquad f_y = x^2 - \frac{1}{3}(3y^2) = x^2 - y^2 = 0$$

[Note that we never have f_x or f_y undefined.]

So $f_y = 0$ tells us that $x^2 = y^2$, so $y = x$ <u>or</u> $y = -x$. We treat each case separately:

If $y = x$, then $f_x = 0$ tells us that $10x + 2xy - 12 = 10x + 2x^2 - 12 = 0$, so

$2x^2 + 10x - 12 = 2(x^2 + 5x - 6) = 2(x+6)(x-1) = 0$, so $x = 1$ <u>or</u> $x = -6$.

This gives the critical points $(1,1)$ and $(-6,-6)$.

If $y = -x$,then $f_x = 0$ tells us that $10x+2xy-12 = 10x+2x(-x)-12 = 10x-2x^2-12 = 0$, so

$2x^2 - 10x + 12 = 2(x^2 - 5x + 6) = 2(x-2)(x-3) = 0$, so $x = 2$ <u>or</u> $x = 3$.

This gives the critical points $(2,-2)$ and $(3,-3)$.

So the critical points of f are $(-6,-6)$, $(1,1)$, $(2,-2)$ and $(3,-3)$.

To determine their type, we need the second partial derivatives and the discriminant:

$$f_{xx} = 10 + 2y \ , \qquad f_{yy} = -2y \ , \qquad \text{and } f_{xy} = 2x$$

So for each of the critical points we have:

At $(-6,-6)$: $f_{xx} = -2 < 0$, $f_{yy} = 12$, $f_{xy} = -12$, $D = (-2)(12) - (-12)^2 = -120 < 0$,
so $D < 0$ and this is a saddle point.

At $(1,1)$: $f_{xx} = 12 > 0$, $f_{yy} = -2$, $f_{xy} = 2$, $D = (12)(-2) - (2)^2 = -28 < 0$,
so $D < 0$ and this is a saddle point.

At $(2,-2)$: $f_{xx} = 6 > 0$, $f_{yy} = 4$, $f_{xy} = 4$, $D = (6)(4) - (4)^2 = 8 > 0$,
so $f_{xx} > 0$ and $D > 0$, so this is a relative minimum.

At $(3,-3)$: $f_{xx} = 4 > 0$, $f_{yy} = 6$, $f_{xy} = 6$, $D = (4)(6) - (6)^2 = -12 < 0$,
so $D < 0$, so this is a saddle point.

An alternative, but slightly less friendly(?), approach is to use $f_x = 10x + 2xy - 12 = 0$ to write $y = \dfrac{12-10x}{2x} = \dfrac{6-5x}{x}$, so $f_y = 0 = x^2 - y^2$ becomes $x^2 - \dfrac{(6-5x)^2}{x^2} = 0$ [Solve this quartic?! Can you 'see' the solutions $x = 1$, $x = 2$?], so $x^4 - (5x-6)^2 = 0$, so $(x^2)^2 = (5x-6)^2$, so $x^2 = 5x - 6$ <u>or</u> $x^2 = -(5x-6)$. Which leads to the same two quadratics!