The Surface Area of a Torus (i.e, doughnut)

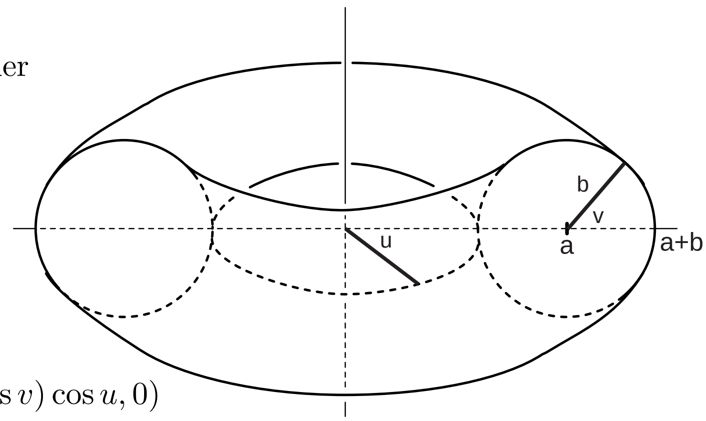
With a parametrization of a torus T_{ab} in hand [a circle of radius b whose center is dragged around a circle of radius a],

$$\begin{aligned}\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) &= ((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v) \\ &= (a\cos u, a\sin u, 0) + (b\cos u\cos v, b\sin u\cos v, b\sin v)\end{aligned}$$

for $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$,

we can compute the stretch factor (Jacobian) for the surface area of the torus and, it turns out, integrate it!

Note that we need to have $a > b$ in order for the torus not to 'run into itself'. This means that down below, when we need to know what $|a + b\cos v|$ is, it is $a + b\cos v$...



Now we compute:

$$\vec{r}_u = (-(a+b\cos v)\sin u, (a+b\cos v)\cos u, 0)$$
$$\vec{r}_v = (-b\sin v\cos u, -b\sin v\sin u, b\cos v)$$

So $\vec{\sigma} = \vec{r}_u \times \vec{r}_v$

$$\begin{aligned}&= ([(a+b\cos v)\cos u][b\cos v] - 0, -([-(a+b\cos v)\sin u][b\cos v] - 0), \\ &\qquad [-(a+b\cos v)\sin u][-b\sin v\sin u] - [(a+b\cos v)\cos u][-b\sin v\cos u]) \\ &= (a+b\cos v)(b\cos u\cos v, b\sin u\cos v, b\sin^2 u\sin v + b\cos^2 u\sin v) \\ &= (a+b\cos v)(b\cos u\cos v, b\sin u\cos v, b\sin v) \\ &= b(a+b\cos v)(\cos u\cos v, \sin u\cos v, \sin v)\end{aligned}$$

[That vector on the right end ought to look familiar; it is how we write points on the unit sphere in spherical coordinates! So the next computation should not be quite so surprising...]

So

$$\begin{aligned}||\vec{\sigma}||^2 &= b^2(a+b\cos v)^2(\cos^2 u\cos^2 v + \sin^2 u\cos^2 v + \sin^2 v) \\ &= b^2(a+b\cos v)^2(\cos^2 v + \sin^2 v) \\ &= b^2(a+b\cos v)^2\end{aligned}$$

So! $||\vec{\sigma}|| = |b(a+b\cos v)| = ab + b^2\cos v$. This is the term we now integrate over the domain of our parametrized surface, $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Which does not require a whole lot of effort on our part! This gives:

$$\begin{aligned}\text{Area}(T_{ab}) &= \int_0^{2\pi}\int_0^{2\pi} ab + b^2\cos v\ dv\ du = \int_0^{2\pi} abv + b^2\sin v\Big|_{v=0}^{v=2\pi} du \\ &= \int_0^{2\pi} (2\pi ab + 0) - (0+0)\ du = \int_0^{2\pi} 2\pi ab\ du = (2\pi ab)(2\pi) \\ &= 4\pi^2 ab\end{aligned}$$