Math 208H
Why Stokes' Theorem is true

Let $\Sigma =$ a region in the xyz-space, parametrized by a region R in the uv-plane.

$$x = x(u,v)\ ,\ y = y(u,v)\ ,\ z = z(u,v)$$

Let $C = \partial R =$ the boundary of R, traversed counterclockwise, parametrized as

$$u = u(t), v = v(t) \quad , \quad a \le t \le b$$

Then the boundary of Σ is a curve E parametrized as

$$x = x(u(t),v(t)), y = y(u(t),v(t)), z = z(u(t),v(t)) \quad , \quad a \le t \le b$$

Let $\vec{F} = \langle L, M, N\rangle =$ a vector field on Σ, and let $\mathrm{curl}(F) = \langle M_z - N_y, -(L_z - N_x), L_y - M_x\rangle$

Then Stokes' Theorem says that

$$(*) \qquad \int\int_\Sigma \mathrm{curl}(\vec{F}) \bullet \vec{N}\ dA = \int_E \vec{F} \bullet\ d\vec{r}$$

To show this, we will translate both integrals to the uv-plane, and use Green's Theorem!

We will start with the path integral. Using the Chain Rule, we can compute that the curve $\vec{r}$ in xyz-space has velocity vector

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle x_u u_t + x_v v_t, y_u u_t + y_v v_t, z_u u_t + z_v v_t\rangle$, so

$$\begin{aligned}\textstyle\int_E \vec{F} \bullet\ dr &= \int_a^b \langle L, M, N\rangle \bullet \langle x_u u_t + x_v v_t, y_u u_t + y_v v_t, z_u u_t + z_v v_t\rangle\ dt \\ &= \int_a^b L\cdot(x_u u_t + x_v v_t) + M\cdot(y_u u_t + y_v v_t) + N\cdot(z_u u_t + z_v v_t)\ dt \\ &= \int_a^b (Lx_u + My_u + Nz_u)u_t + (Lx_v + My_v + Nz_v)v_t\ dt\end{aligned}$$

which, from the point of view of the uv-plane, is the integral of the vector field

$\vec{G} = \langle Lx_u + My_u + Nz_u, Lx_v + My_v + Nz_v\rangle$

over the curve C . (Technical point: L, M , and N are thought of here as functions of u and v, not of x, y and z. So these really stand for $L(x(u,v), y(u,v), z(u,v)$, etc.) But since C is the boundary of the region R, traversed counterclockwise, **Green's Theorem** tells us that

$$\int_C \vec{G} \bullet\ d\vec{r} = \int\int_R \mathrm{curl}(\vec{G})\ dA$$

But, using the Chain Rule,

$$\begin{aligned}\mathrm{curl}(\vec{G}) & \\ &= (Lx_v + My_v + Nz_v)_u - (Lx_u + My_u + Nz_u)_v \\ &= (Lx_v)_u + (My_v)_u + (Nz_v)_u - (Lx_u)_v - (My_u)_v - (Nz_u)_v \\ &= (L_u x_v + Lx_{vu}) + (M_u y_v + My_{vu}) + (N_u z_v + Nz_{vu}) \\ &\qquad -(L_v x_u + Lx_{uv}) - (M_v y_u + My_{uv}) - (N_v z_u + Nz_{uv}) \\ &= L_u x_v - L_v x_u + M_u y_v - M_v y_u + N_u z_v - N_v z_u \\ &\qquad +(Lx_{vu} - Lx_{uv}) + (My_{vu} - My_{uv}) + (Nz_{vu} - Nz_{uv}) \\ &= L_u x_v - L_v x_u + M_u y_v - M_v y_u + N_u z_v - N_v z_u\end{aligned}$$

because mixed partials are equal. So

$$\int_E \vec{F}\bullet\ dr = \int_C \vec{G}\bullet\ d\vec{r} = \int\int_R L_u x_v - L_v x_u + M_u y_v - M_v y_u + N_u z_v - N_v z_u\ du\ dv$$

So to show that Stokes' Theorem is true, it is enough to show that

$$\int\int_\Sigma \text{curl}(\vec{F}) \bullet \vec{N}\ dA$$

is equal to the integral over R above. But! Using our parametrization of Σ,

$$\int\int_\Sigma \text{curl}(\vec{F}) \bullet \vec{N}\ dA = \int\int_R \text{curl}(\vec{F}) \bullet (T_u \times T_v)\ du\ dv$$

and so it is enough to show that

$$\text{curl}(\vec{F}) \bullet (T_u \times T_v) = L_u x_v - L_v x_u + M_u y_v - M_v y_u + N_u z_v - N_v z_u$$

But!

$\text{curl}(\vec{F}) = \langle N_y - M_z, -(N_x - L_z), M_x - L_y\rangle$, and

$T_u = \langle x_u, y_u, z_u\rangle$, $T_v = \langle x_v, y_v, z_v\rangle$, so $T_u \times T_v = \langle y_u z_v - y_v z_u, -(x_u z_v - x_v z_u), x_u y_v - x_v y_u\rangle$

and so

$$\begin{aligned}
&\text{curl}(\vec{F}) \bullet (T_u \times T_v)\\
&= \langle N_y - M_z, -(N_x - L_z), M_x - L_y\rangle \bullet \langle y_u z_v - y_v z_u, -(x_u z_v - x_v z_u), x_u y_v - x_v y_u\rangle\\
&= (N_y - M_z)(y_u z_v - y_v z_u) + (N_x - L_z)(x_u z_v - x_v z_u) + (M_x - L_y)(x_u y_v - x_v y_u)\\
&= N_y y_u z_v - N_y y_v z_u - M_z y_u z_v + M_z y_v z_u + N_x x_u z_v - N_x x_v z_u\\
&\qquad - L_z x_u z_v + L_z x_v z_u + M_x x_u y_v - M_x x_v y_u - L_y x_u y_v + L_y x_v y_u\\
&= (L_z z_u + L_y y_u) x_v + (M_x x_u + M_z z_u) y_v + (N_y y_u + N_x x_u) z_v\\
&\qquad - (L_y y_v + L_z z_v) x_u - (M_x x_v + M_z z_v) y_u - (N_x x_v + N_y y_v) z_u\\
&= (L_y y_u + L_z z_u) x_v + (M_x x_u + M_z z_u) y_v + (N_x x_u + N_y y_u) z_v\\
&\qquad + (L_x x_u x_v + M_y y_u y_v + N_z z_u z_v)\\
&\qquad - (L_y y_v + L_z z_v) x_u - (M_x x_v + M_z z_v) y_u - (N_x x_v + N_y y_v) z_u\\
&\qquad - (L_x x_u x_v + M_y y_u y_v + N_z z_u z_v)\\
&= (L_y y_u + L_z z_u) x_v + (M_x x_u + M_z z_u) y_v + (N_x x_u + N_y y_u) z_v\\
&\qquad + (L_x x_u x_v + M_y y_u y_v + N_z z_u z_v)\\
&\qquad - (L_y y_v + L_z z_v) x_u - (M_x x_v + M_z z_v) y_u - (N_x x_v + N_y y_v) z_u\\
&\qquad - (L_x x_u x_v + M_y y_u y_v + N_z z_u z_v)\\
&= (L_x x_u + L_y y_u + L_z z_u) x_v + (M_x x_u + M_y y_u + M_z z_u) y_v\\
&\qquad + (N_x x_u + N_y y_u + N_z z_u) z_v\\
&\qquad - (L_x x_v + L_y y_v + L_z z_v) x_u - (M_x x_v + M_y y_v + M_z z_v) y_u\\
&\qquad - (N_x x_v + N_y y_v + N_z z_v) z_u\\
&= L_u x_v + M_u y_v + N_u z_v - L_v x_u - M_v y_u - N_v z_u\\
&= L_u x_v - L_v x_u + M_u y_v - M_v y_u + N_u z_v - N_v z_u
\end{aligned}$$

as desired!

Note that we used the Chain Rule again, showing that $L_x x_u + L_y y_u + L_z z_u = L_u$, etc.

Which is why Stokes' Theorem is true....