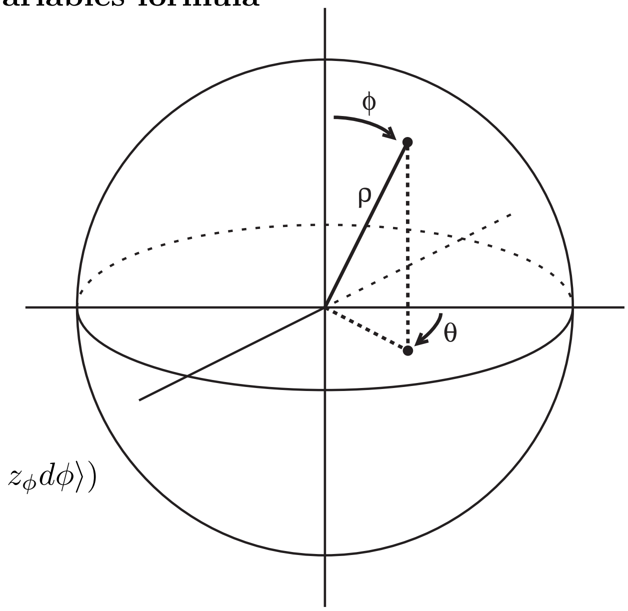
Math 208H

Spherical coordinates : the change of variables formula



To compute a triple integral in spherical coordinates (e.g., to integrate over the inside of a sphere...), we need to work out its Jacobian determinant. That is, for

$$x = (\rho\sin\phi)\cos\theta = \rho\cos\theta\sin\phi$$
$$y = (\rho\sin\phi)\sin\theta = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

we need to compute

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \langle x_\rho d\rho, y_\rho d\rho, z_\rho d\rho\rangle \bullet (\langle x_\theta d\theta, y_\theta d\theta, z_\theta d\theta\rangle \times \langle x_\phi d\phi, y_\phi d\phi, z_\phi d\phi\rangle)$$
$$= \langle x_\rho, y_\rho, z_\rho\rangle \bullet (\langle x_\theta, y_\theta, z_\theta\rangle \times \langle x_\phi, y_\phi, z_\phi\rangle) d\rho d\theta d\phi$$

And so we compute:

$$\langle x_\rho, y_\rho, z_\rho\rangle \bullet (\langle x_\theta, y_\theta, z_\theta\rangle \times \langle x_\phi, y_\phi, z_\phi\rangle)$$
$$= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \bullet (\langle\rho\sin\theta\sin\phi, -\rho\cos\theta\sin\phi, 0\rangle \times \langle\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, -\rho\sin\phi\rangle)$$
$$= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \bullet \langle\rho^2\cos\theta\sin^2\phi - 0, -(0-\rho^2\sin\theta\sin^2\phi), \rho^2\sin^2\theta\sin\phi\cos\phi + \rho^2\cos^2\theta\sin\phi\cos\phi\rangle$$
$$= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \bullet \langle\rho^2\cos\theta\sin^2\phi, \rho^2\sin\theta\sin^2\phi, \rho^2(\sin^2\theta+\cos^2\theta)\sin\phi\cos\phi\rangle$$
$$= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \bullet \langle\rho^2\cos\theta\sin^2\phi, \rho^2\sin\theta\sin^2\phi, \rho^2\sin\phi\cos\phi\rangle$$
$$= \rho^2\cos^2\theta\sin^3\phi + \rho^2\sin^2\theta\sin^3\phi + \rho^2\sin\phi\cos^2\phi$$
$$= \rho^2(\sin^2\theta+\cos^2\theta)\sin^3\phi + \rho^2\sin\phi\cos^2\phi$$
$$= \rho^2\sin^3\phi + \rho^2\sin\phi\cos^2\phi$$
$$= \rho^2\sin\phi(\sin^2\phi+\cos^2\phi)$$
$$= \rho^2\sin\phi$$

And so when we compute a triple integral using spherical coordinates, we need to include the "fudge factor"

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\, d\rho\ d\theta\ d\phi = \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$