A Change of Variables Formula for Double Integrals:
The Successor to u-Substitution

With one variable, we have an integration technique we call u-substitution;

if we set $u = g(x)$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} g(u)\,du$.

We can interpret this in a way that will help us generalize it to several variables. The function g carries the interval $[a, b]$ to the interval $[g(a), g(b)]$, and we can imagine computing the integral $\int_{g(a)}^{g(b)} g(u)\,du$ from Riemann sums, but choosing the points to cut the interval $[g(a), g(b)]$ into using $[a, b]$, by choosing $a = x_0 < x_1 < \cdots < x_n = b$ and using the points $g(a) = g(x_0) < g(x_1) < \cdots < g(x_n) = g(b)$. If we call $g(x_i) = u_i$, then $\int_{g(a)}^{g(b)} f(u)\,du$ is approximated by

$$\begin{aligned}\sum f(u_i^*)\Delta u_i = \sum f(u_i^*)(u_{i+1} - u_i) = \sum f(g(x_i^*))(g(x_{i+1}) - g(x_i))\\ \approx \sum f(g(x_i^*))g'(x_i^*)(x_{i+1} - x_i) = \sum f(g(x_i^*))g'(x_i^*)\Delta x_i\end{aligned}$$

where the approximation in the middle comes from the Mean Value Theorem. But this last sum is an approximation for $\int_a^b f(g(x))g'(x)\,dx$, so this last sum gives approximations to both integrals, implying that the two integrals are equal!

The same idea can be applied to functions of several variables. The idea above is that we write our Reimann sum for $\int_{g(a)}^{g(b)} f(u)\,du$ not using evenly-spaced points, but rather using points determined by the function g. We can do the same thing with several variables, but instead of using rectangles we use shapes determined by a function g, or rather, the images of rectangles under the function g. Unlike for a single variable, though, we need to be a little careful that our *change of variables* function g does not backtrack (like a u-substitution function could); more about this later.

The general idea is that if a region R can be described more conveniently using a different sort of coordinates, this means that we are describing x and y as *functions* of different variables s and t. For example, a circle of radius 4 is a 'polar' rectangle:

$$x = r\cos\theta \text{ and } y = r\sin\theta, \text{ for } 0 \le r \le 4 \text{ and } 0 \le \theta \le 2\pi$$

(i.e., polar coordinates). In general, changing coordinates means describing the region R by

$$x = x(s,t) \text{ and } y = y(s,t), \text{ for } s \text{ and } t \text{ in some region } S$$

That is, we are using a function $g(s,t) = (x(s,t), y(s,t))$, for a function $g : S \to R$, to describe the points in R in terms of the points in S. Then we write the integral of the function f over R as the integral of *something else* (written in terms of s and t) over the region S. The question is, the integral of *what*?

The answer, just like u-substitution, comes from thinking of cutting up S into little rectangles S_{ij}, and looking at the little regions R_{ij} that the change of variables function g carries each to; these regions R_{ij} are a way to cut R into little pieces, which we can use to create a Rieman sum. The integral of f over R can be approximated by adding up chosen values over each region R_{ij}, times the area of R_{ij}. By choosing (s_i, t_j) in S_{ij}, we can use

f evaluated at the point $(x(s_i,t_j),y(s_i,t_j))$ in R_{ij} as the height of our parallelopiped; the real question is, what is the area of R_{ij}?.

If we think of the rectangles S_{ij} as having sides of (small) length ds and dt, then using linear approximations to $x(s,t)$ and $y(s,t)$, R_{ij} can be thought of as being approximately a parallelogram with sides the vectors

$$(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s})ds \text{ and } (\frac{\partial x}{\partial t},\frac{\partial y}{\partial t})dt$$

(at least, its area will be approximated by the area of this parallelogram). But we know how to compute the area of such a parallelogram! It is given by the length of the cross product of the two sides (adding 0's to the vectors, so they are in 3-space!), which turns out to be:

$$\Delta A_{ij} = |x_s y_t - x_t y_s| \ ds \ dt$$

So the Reimann sum $\sum f(x_i,y_j)\text{Area}(R_{ij})$ can be approximated by

$$\sum f(x(s_i,t_j),y(s_i,t_j))|x_s y_t - x_t y_s| \ \text{Area}(S_{ij}) \ .$$

Taking limits as the size of the S_{ij} (i.e, ds and dt) goes to zero, we then obtain:

$$\int\int_R f(x,y) \ dx \ dy = \int\int_S f(x(s,t),y(s,t)) \ |x_s y_t - x_t y_s| \ ds \ dt$$

This is our change of variables formula for double integrals. The expression $|x_s y_t - x_t y_s|$ is called the *Jacobian* (or *Jacobian determinant*) associated to the change of variables, and is sometimes written

$$|x_s y_t - x_t y_s| = \frac{\partial(x,y)}{\partial(s,t)}$$

For example, to integrate a function f over the triangle with vertices (1,1), (2,3), and (3,8), we can instead integrate over the triangle with vertices (0,0), (1,0), and (0,1), by changing coordinates. It turns out we can always do this by writing

$$x = as + bt + c \text{ and } y = ds + et + f$$

for appropriate choices of a,b,c,d,e and f. All you need to do is solve the equations $1 = a0 + b0 + c, 1 = d0 + e0 + f, 2 = a1 + b0 + c, 3 = d1 + e0 + f, 3 = a0 + b1 + c$, and $8 = d0 + e1 + f$ which, in this case, gives a=1,b=2,c=1,d=2,e=7,f=1. So $x = s + 2t + 1$ and $y = 2s + 7t + 1$. This means that the function $g(s,t) = (s + 2t + 1, 2s + 7t + 1)$ carries the triangle S with vertices (0,0), (1,0), and (0,1) to the triangle R with vertices (1,1), (2,3), and (3,8), enabling us to compute integrals over R by writing them as integrals over S instead.

In this case, we compute that the Jacobian is $|1 \cdot 7 - 2 \cdot 2| = |3| = 3$. So under this change of coordinates,

$$\int\int_R f(x,y) \ dA = \int_0^1 \int_0^{1-t} f(s + 2t + 1, 2s + 7t + 1) \cdot 3 \ ds \ dt$$

This (depending on f !) will probably be a better integral to compute than trying to express the integral over R directly as an iterated integral...

Probably our most popular change of variable will be polar coordinates: the function $g(r,\theta) = (r\cos\theta, r\sin\theta)$ carries rectangles to interesting shapes (circles, semicircles, pie sectors). In this case the Jacobian 'correction factor' is

$$|(\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta)| = |r| = r \ ,$$

allowing us to compute $\int\int_R f(x,y) \ dx \ dy$ as $\int\int_S f(r\cos\theta, r\sin\theta) \ r \ dr \ d\theta$, instead.