Math 208H Exam 2 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

Some potentially useful formulas can be found at the bottom of the last page.

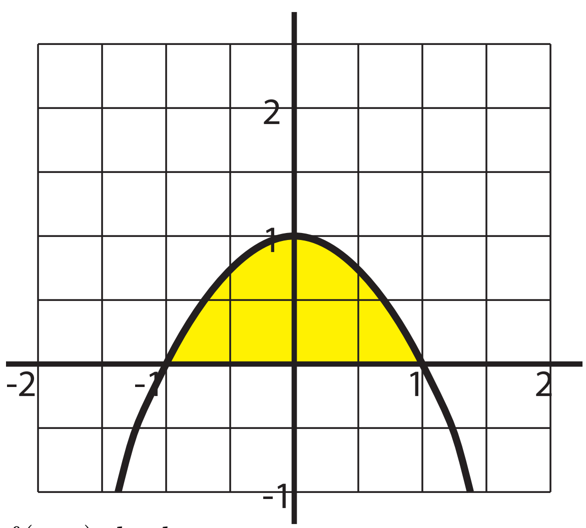
1. (20 pts.) What point along the (level) curve $g(x,y) = x^2 + y^2 - 3x + xy = 0$ has the largest y-coordinate?

The function to maximize is $f(x,y) = y$, subject to the constraint $g(x,y) = x^2 + y^2 - 3x + xy = 0$. We compute:
$\nabla f = (0,1)$, $\nabla g = (2x - 3 + y, 2y + x)$, so we wish to solve
$0 = \lambda(2x - 3 + y)$, $1 = \lambda(2y + x$, and $x^2 + y^2 - 3x + xy = 0$.

The first requires $\lambda = 0$ or $2x - 3 + y = 0$, but if $\lambda = 0$ then the second equation reads $1 = 0$, which is impossible. So we must have $2x - 3 + y = 0$, so $y = 3 - 2x$. Plugging this into the third equation gives
$0 = x^2 + y^2 - 3x + xy = x^2 + (3 - 2x)^2 - 3x + x(3 - 2x)$
$= x^2 + (9 - 12x + 4x^2) - 3x + (3x - 2x^2) = 9 - 12x + 3x^2 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$, so either $x = 1$ or $x = 3$. Then $y = 3 - 2x$ tells us that $y = 1$ or $y = -3$. Since $y = 1$ is largest, we find that the largest y-value occurs at $(1,1)$.

2. (20 pts.) Sketch the region R over which the integral $\int\int_R f(x,y)\ dA$ can computed by the iterated integral $\int_{-1}^{1} \int_{0}^{1-x^2} f(x,y)\ dy\ dx$, and express this integral as an iterated integral with the opposite order of integration.

The region R is described as
$0 \le y \le 1 - x^2$ for $-1 \le x \le 1$,
so it is the region lying under the graph of $y = 1 - x^2$ between -1 and 1.
Writing this in terms of horizontal lines, we compute:
$y = 1 - x^2$, so $x^2 = 1 - y$, so $x = \pm\sqrt{1-y}$.
The lowest and highest horizontal lines that hit the region R are $y = 0$ and $y = 1$.
So the region can also be described as
$-\sqrt{1-y} \le x \le \sqrt{1-y}$ for $0 \le y \le 1$.



This gives the iterated integral $\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\ dx\ dy$.

3. (20 pts.) Find the integral of the function $f(x,y) = xy$ over the region R lying to the right of the y-axis, above the parabola $y = x^2 - 2$, and below the line $y = 4 - x$.

Integrating $dy\ dx$, we need to find the rightmost vertical line that hits the region, which is where $y = x^2 - 2$ and $y = 4 - x$ meet. So we compute:

$x^2 - 2 = 4 - x$ when $0 = x^2 - 2 - 4 + x = x^2 + x - 6 = (x+3)(x-2)$, so $x = -3$ or $x = 2$. Since we are working to the right of the y-axis, our intersection occurs at $x = 2$. So our iterated integral is

$$\int_0^2 \int_{x^2-2}^{4-x} xy\ dy\ dx = \int_0^2 x\frac{y^2}{2}\Big|_{y=x^2-2}^{y=4-x} dx = \int_0^2 x\frac{(x^2-2)^2}{2} - x\frac{(4-x)^2}{2}\ dx$$

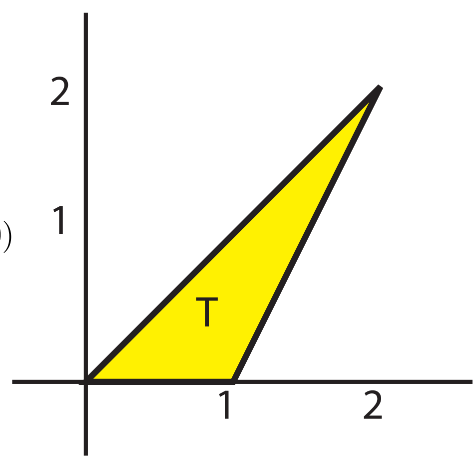
$$= \int_0^2 x\frac{x^2 - 8x + 16}{2} - x\frac{x^4 - 4x^2 + 4}{2}\ dx = \int_0^2 \frac{1}{2}x^3 - 4x^2 + 8x - \frac{1}{2}x^5 + 2x^3 - 2x\ dx$$

$$= \int_0^2 -\frac{1}{2}x^5 + \frac{5}{2}x^3 - 4x^2 + 6x\ dx = -\frac{1}{12}x^6 + \frac{5}{8}x^4 - \frac{4}{3}x^3 + 3x^2\Big|_0^2 = -\frac{64}{12} + \frac{80}{8} - \frac{32}{3} + 12$$

$$= -\frac{16}{3} + 10 - \frac{32}{3} + 12 = -\frac{48}{3} + 22 = -16 + 22 = 6$$

4. (20 pts.) Find the average value of the function $f(x,y) = y$ over the triangle T with vertices $(0,0)$, $(1,0)$ and $(2,2)$. [Straight at it or change of variables, your choice!]

The region can be integrated over either $dy\ dx$ or $dx\ dy$; the second one does not require cutting into pieces. We need the equations for the lines; the upper one is $y = x$ (since it passes through $(0,0)$ and $(2,2)$, and the lower one is $y = 2(x-1) = 2x - 2$ (since it passes through $(1,0)$ with a slope of 2). This becomes $x = y$ on the left and $x = 1 + y/2$ on the right. So



$$\int\int_T y\ dA = \int_0^2 \int_y^{1+y/2} y\ dx\ dy$$

$$= \int_0^2 xy\Big|_y^{1+y/2} dy = \int_0^2 (1 + y/2)y - y^2\ dy$$

$$= \int_0^2 y + y^2/2 - y^2\ dy = \int_0^2 y - y^2/2\ dy = \frac{1}{2}y^2 - \frac{1}{6}y^3\Big|_0^2 = (2 - \frac{8}{6}) - (0 - 0) = 2 - \frac{4}{3} = \frac{2}{3}$$

The average value is this number <u>divided</u> <u>by</u> <u>the</u> <u>area</u> of T. But T is a triangle with base $b = 1$ and height $h = 2$, so Area $= \frac{1}{2}(1)(2) = 1$ (!), so the average value of f over T is $(2/3)/1 = 2/3$.

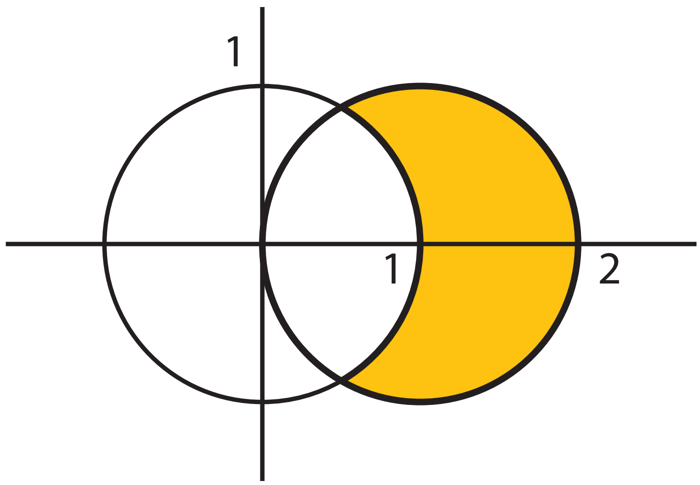
OR! We can apply a change of coordinates $x = au + bv + c$ and $y = du + ev + f$ sending (say) the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ to T. We then find that $(0,0) = (c,f)$, so $c = f = 0$; $(1,0) = (a+c, d+f) = (a,d)$, so $a = 1$ and $d = 0$; and finally $(2,2) = (a+b+c, d+e+f) = (1+b, e)$, so $b = 1$ and $e = 2$. So our change of variables is $x = u + v$ and $y = 2v$. Then the Jacobian is $J = |x_u y_v - x_v y_u| = |1 \cdot 2 - 1 \cdot 0| = |2| = 2$.

Then since our uv-triangle can be described as $0 \le v \le u$ for $0 \le u \le 1$, we have $\int\int_T y\ dA = \int_0^1 \int_0^u (2v)J\ dv\ du = \int_0^1 \int_0^u 4v\ dv\ du = \int_0^1 2v^2\Big|_0^u\ du = \int_0^1 2u^2 - 0\ du = \int_0^1 2u^2\ du = \frac{2}{3}u^3\Big|_0^1 = \frac{2}{3}1^3 - 0 = \frac{2}{3}$.

And since, again, the area of T is 1, we have that the average value of $f(x,y) = y$ over the region T is $(2/3)/1 = 2/3$.

5. (20 pts.) Find the area of the *lune*: the region described, in polar coordinates, as lying outside of the circle $r = 1$ and inside of the circle (centered at $(x,y) = (1,0)$) $r = 2\cos\theta$ (see figure). [Hint: At what (polar) angles do the two circles meet?]

The two circles meet when $1 = r = 2\cos\theta$, so $\cos\theta = \frac{1}{2}$. This happens when $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3}$. Then the lune can be described, in polar coords, as $1 \le r \le 2\cos\theta$ for $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$. We can therefore compute the area using polar coords:



$$\text{Area} = \int\int_L 1\ dA$$

$$= \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r\ dr\ d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{r^2}{2}\Big|_1^{2\cos\theta}\ d\theta = \int_{-\pi/3}^{\pi/3} 2\cos^2\theta - \frac{1}{2}\ d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 2(\frac{1}{2}(1+\cos(2\theta))) - \frac{1}{2}\ d\theta = \int_{-\pi/3}^{\pi/3} \cos(2\theta) + \frac{1}{2}\ d\theta = \frac{1}{2}\sin(2\theta) + \frac{1}{2}\theta\Big|_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}[(\sin(2\pi/3) + \pi/3) - (-\sin(2\pi/3) - \pi/3)] = \frac{1}{2}(2)(\frac{\sqrt{3}}{2} + \frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Some potentially useful formulas:

$\sin^2 x = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2 x = \frac{1}{2}(1 + \cos(2x))$