Math 208H, Section 1

Practice problems for Exam 1 (Solutions)

[**Disclaimer:** these solutions were written somewhat hastily and without much verification, so while the method described is almost certainly correct, the actual computations do not carry the same claims of correctness....]

1. Find the **sine** of the angle between the vectors (1,–1,2) and (1,2,1) .

We can use the dot product (dividing by lengths) to compute the cosine of the angle, and then from that the sine. Or we can use $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$ to compute the sine, by finding the cross product and computing lengths.
$\sin(\theta) = \sqrt{(-5)^2+1^2+3^2}/(\sqrt{1^2+(-1)^2+2^2}\cdot\sqrt{1^2+2^2+1^2}) = \sqrt{35}/(\sqrt{6}\cdot\sqrt{6}) = \sqrt{35}/6$
This is consistent with $\cos(\theta) = (1\cdot 1+(-1)\cdot 2+2\cdot 1)/(\sqrt{6}\cdot\sqrt{6}) = 1/6$.

2. Show that if the vectors $\vec{\mathbf{v}}=(a_1,a_2,a_3)$ and $\vec{\mathbf{w}}=(b_1,b_2,b_3)$ have the same length, then the vectors $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ are perpendicular to one another.

We wish to know that $(\vec{\mathbf{v}}+\vec{\mathbf{w}})\circ(\vec{\mathbf{v}}-\vec{\mathbf{w}})=0$. But expanding this out, we find that it is equal to $\vec{\mathbf{v}}\circ\vec{\mathbf{v}}-\vec{\mathbf{w}}\circ\vec{\mathbf{w}}$. This will be equal to 0 precisely when $|\vec{v}|^2 = \vec{\mathbf{v}}\circ\vec{\mathbf{v}} = \vec{\mathbf{w}}\circ\vec{\mathbf{w}} = |\vec{w}|^2$. This in turn, means that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same length.

3. Find the equation of the plane in 3-space which passes through the three points $(1,2,1)$, $(6,1,2)$, and $(9,-2,1)$. Does the point $(3,2,1)$ lie on this plane?

To find the equation, we need a point and a normal vector; the normal can be found by a cross product. $\vec{N} = \vec{PQ}\times\vec{PR} = (5,-1,1)\times(8,-4,0) = (4,8,-12)$. Then the equation is $(4,8,-12)\circ(x-1,y-2,z-1)=0$, or $4x+8y-12z=8$ (or $x+2y-3z=2$ (!)). [Check: the 3 points satisfy the equation!] Checking, $4\cdot 3+8\cdot 2-12\cdot 1 = 16\neq 8$, so the point does not lie on the plane.

4. Find the partial derivatives of the following functions:

(a) $f(x,y,z) = x\tan(2x+yz)$

We have $f_x = \tan(2x+yz)+x\sec^2(2x+yz)\cdot 2$, $f_y = x\sec^2(2x+yz)\cdot z$, and $f_z = x\sec^2(2x+yz)\cdot y$.

(b) $g(x,y) = \dfrac{x^2y-ty^4}{\sin(3y)+4}$ We have $g_x = \dfrac{(2xy)(\sin(3y)+4)-(x^2y-ty^4)(0)}{(\sin(3y)+4)^2}$,

and $g_y = \dfrac{(x^2-4ty^3)(\sin(3y)+4)-(x^2y-ty^4)(3\cos(3y))}{(\sin(3y)+4)^2}$. Since the question didn't ask us to do anything with these, why simplify them?

5. Find the equation of the tangent plane to the graph of the equation $f(x,y,z) = xy^2+x^2z-xyz = 5$, at the point $(-1,1,3)$.

$f_x = y^2+2xz-yz$, $f_y = 2xy+x^2-xz$, and $f_z = x^2-xy$. The normal vector to the plane will be $(f_x(-1,1,3), f_y(-1,1,3), f_z(-1,1,3)) = (1-6-3, -2+1+3, 1+1) = (-8,2,2)$. Together with the point of tangency, this gives us the equation

$-8(x-(-1))+2(y-1)+2(z-3)=0$, or $-8x+2y+2z=16$, or $4x-y-z=-8$.

6. Calculate the first and second partial derivatives of the function $h(x,y) = \dfrac{\sin(x+y)}{y}$

It may help a bit to write this function as $h(x,y) = y^{-1}\sin(x+y)$. Then we have
$h_x = y^{-1}\cos(x+y)\cdot 1 = y^{-1}\cos(x+y)$,
$h_y = -y^{-2}\sin(x+y) + y^{-1}\cos(x+y)\cdot 1 = -y^{-2}\sin(x+y) + y^{-1}\cos(x+y)$. Then
$h_{xx} = (h_x)_x = y^{-1}(-\sin(x+y)\cdot 1) = -y^{-1}\sin(x+y)$
$h_{yx} = h_{xy} = (h_x)_y = -y^{-2}\cos(x+y) + y^{-1}(-\sin(x+y)\cdot 1)$
$= -y^{-2}\cos(x+y) - y^{-1}\sin(x+y)$
$h_{yy} = (h_y)_y$
$= [2y^{-3}\sin(x+y) - y^{-2}(\cos(x+y)\cdot 1)] + [-y^{-2}\cos(x+y) + y^{-1}(-\sin(x+y)\cdot 1)]$
Again, we don't want to do anything with it, so why bother simplifying it...

7. In which direction is the function $f(x,y) = x^4y - 3x^2y^2$ increasing the fastest, at the point (1,2) ? In which directions is the function *neither* increasing *nor* decreasing?

f increases fastest in the direction of the gradient, so we compute:
$\nabla f = (4x^3y - 6xy^2, x^4 - 6x^2y)$, which at (1,2) gives $\vec{v} = (8-24, 1-12) = (-16,-11)$. This is the drection of fastest increase (you can divide by its length if you want a unit vector...).

For no increase/decrease, what we want is $D_{\vec{w}}f = \nabla f \circ \vec{w} = 0$, so we want
$(-16,11)\circ(\alpha,\beta) = -16\alpha - 11\beta = 0$; we can do this, for example, with $\vec{w} = (\alpha,\beta) = (11,-16)$. [There are many other answers, all scalar multiples of this one.]

8. If $f(x,y) = x^2y^5 - x + 3y - 4$, $x = x(u,v) = \dfrac{u}{u+v}$ and $y = y(u,v) = uv - u$, use the Chain Rule to find $\dfrac{\partial f}{\partial u}$ when $u = 1$ and v=0 .

First, when $(u,v) = (1,0)$, then $x = 1/(1+0) = 1$ and $y = 1\cdot 0 - 1 = -1$. From the chain rule, we know that $f_u = f_x x_u + f_y y_u$, evaluated at $(x,y) = (1,-1)$ and $(u,v) = (1,0)$. We compute:

$f_x = 2xy^5 - 1 = -2-1 = -3$, $f_y = 5x^2y^4 + 3 = 5+3 = 8$, $x_u = \dfrac{(1)(u+v)-(u)(1)}{(u+v)^2} = \dfrac{v}{(u+v)^2} = 0$, and $y_u = v - 1 = 0 - 1 = -1$; so at $(u,v) = (1,0)$ we have $f_u(1,0) = (-3)(0) + (8)(-1) = -8$.

9. Find the local extrema of the function $f(x,y) = 2x^4 - 2xy + y^2$, and determine, for each, if it is a local max. local min, or saddle point.

Local extrema occur at critical points, so we compute: $f_x = 8x^3 - 2y$ and $f_y = -2x + 2y$. These are never undefined, so our only critical points will occur when both are 0. $f_y = -2x + 2y = 0$ means $2y = 2x$, so $y = x$. Substituting this into $f_x = 8x^3 - 2y = 0$ gives $8x^3 - 2x = (2x)(4x^2 - 1) = 0$, so either $x = 0$, or $4x^2 - 1 = 0$, so $x = 0$ or $x = 1/2$ or $x = -1/2$. This yields the three critical points $(0,0)$, $(1/2,1/2)$, and $(-1/2,-1/2)$.

To determine their character, we need the Hessian: $f_{xx} = 24x^2$, $f_{xy} = -2$, and $f_{yy} = 2$, so $H = f_{xx}f_{yy} - (f_{xy})^2 = 48x^2 - 4$. At $(0,0)$ $H = -4 < 0$, so $(0,0)$ is a saddle point. At $(1/2,1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$, so $(1/2,1/2)$ is a local min. And at $(-1/2,-1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$ as well, so $(-1/2,-1/2)$ is also a local min.