A reduction formula for $\int_0^{2\pi} \sin^{2n} x \; dx$

For notation, set $A_n = \int_0^{2\pi} \sin^{2n} x \; dx$. Then:

$$\begin{aligned}
A_n &= \int_0^{2\pi} \sin^{2n} x \; dx = \int_0^{2\pi} \sin^{2n-2} x \sin^2 x \; dx = \int_0^{2\pi} \sin^{2n-2} x(1-\cos^2 x) \; dx \\
&= \int_0^{2\pi} \sin^{2n-2} x \; dx - \int_0^{2\pi} \sin^{2n-2} x \cos^2 x \; dx = A_{n-1} - \int_0^{2\pi} \sin^{2n-2} x \cos^2 x \; dx \\
&= A_{n-1} - \int_0^{2\pi} (\sin^{2n-3} x \cos x)(\sin x \cos x) \; dx
\end{aligned}$$

Now, make a u-substitution:

$u = \sin^{2n-3} x \cos x$, so $du = (2n-3)\sin^{2n-4} x \cos^2 x - \sin^{2n-2} x \; dx$;

$dv = \sin x \cos x$, so $v = \frac{1}{2}\sin^2 x$

Then:

$$\begin{aligned}
A_n &= A_{n-1} - \int_0^{2\pi} u \; dv = A_{n-1} - [uv \Big|_0^{2\pi} - \int_0^{2\pi} v \; du] \\
&= A_{n-1} - [\left(\frac{1}{2}\sin^2 x\right)\left\{\sin^{2n-3} x \cos^2 x\right\} \Big|_0^{2\pi} \\
&\qquad - \int_0^{2\pi} \left(\frac{1}{2}\sin^2 x\right)\left\{(2n-3)\sin^{2n-4} x \cos^2 x - \sin^{2n-2} x\right\} \; dx] \\
&= A_{n-1} - \left((0-0) - \left(\int_0^{2\pi} \frac{2n-3}{2}\sin^{2n-2} x \cos^2 x \; dx - \frac{1}{2}\int_0^{2\pi} \sin^{2n} x \; dx\right)\right) \\
&= A_{n-1} + \frac{2n-3}{2}\int_0^{2\pi} \sin^{2n-2} x \cos^2 x \; dx - \frac{1}{2}A_n \\
&= A_{n-1} - \frac{1}{2}A_n + \frac{2n-3}{2}\int_0^{2\pi} \sin^{2n-2} x \cos^2 x \; dx \\
&= A_{n-1} - \frac{1}{2}A_n + \frac{2n-3}{2}\int_0^{2\pi} \sin^{2n-2} x \left(1-\sin^2 x\right) \; dx \\
&= A_{n-1} - \frac{1}{2}A_n + \frac{2n-3}{2}\int_0^{2\pi} \sin^{2n-2} x \; dx - \frac{2n-3}{2}\int_0^{2\pi} \sin^{2n} x \; dx \\
A_n &= A_{n-1} - \frac{1}{2}A_n + \frac{2n-3}{2}A_{n-1} - \frac{2n-3}{2}A_n = \frac{2n-1}{2}A_{n-1} - (n-1)A_n
\end{aligned}$$

So: $nA_n = \frac{2n-1}{2}A_{n-1}$, so

$$A_n = \frac{2n-1}{2n}A_{n-1} \quad \text{, i.e.,}$$

$$\int_0^{2\pi} \sin^{2n} x \; dx = \frac{2n-1}{2n}\int_0^{2\pi} \sin^{2n-2} x \; dx$$

So, since $$A_0 = \int_0^{2\pi} \sin^0 x\ dx = \int_0^{2\pi} 1\ dx = 2\pi\ ,$$

$$A_1 = \int_0^{2\pi} \sin^2 x\ dx = \frac{1}{2}A_0 = \pi$$

$$A_2 = \int_0^{2\pi} \sin^4 x\ dx = \frac{3}{4}A_1 = \frac{3}{4}\pi$$

$$A_3 = \int_0^{2\pi} \sin^6 x\ dx = \frac{5}{6}A_2 = \frac{5}{8}\pi$$

$$A_4 = \int_0^{2\pi} \sin^8 x\ dx = \frac{7}{8}A_3 = \frac{35}{64}\pi$$

$$A_5 = \int_0^{2\pi} \sin^{10} x\ dx = \frac{9}{10}A_4 = \frac{63}{128}\pi$$

$$A_6 = \int_0^{2\pi} \sin^{12} x\ dx = \frac{11}{12}A_5 = \frac{231}{512}\pi$$

$$A_7 = \int_0^{2\pi} \sin^{14} x\ dx = \frac{13}{14}A_6 = \frac{429}{1024}\pi$$

$$A_8 = \int_0^{2\pi} \sin^{16} x\ dx = \frac{15}{16}A_7 = \frac{6435}{16384}\pi$$

$$A_9 = \int_0^{2\pi} \sin^{18} x\ dx = \frac{17}{18}A_8 = \frac{12155}{32768}\pi$$

$$A_{10} = \int_0^{2\pi} \sin^{20} x\ dx = \frac{19}{20}A_9 = \frac{46189}{131072}\pi$$

And we could of course keep going, but you get the idea.....

The integrals from 0 to π are half of these values, and of course

$$\int_0^{2\pi} \sin^{2n-1} x\ dx = \int_0^{\pi} \sin^{2n-1} x\ dx = 0$$

And, finally, since $\cos x$ is really just $\sin x$ shifted by $\frac{\pi}{2}$ (to the left), and we are integrating over an entire period of $\sin x$, the integrals for the powers of $\cos x$ are exactly the same!