Math 189H Joy of Numbers Activity Log

Tuesday, September 27, 2011

John Blasik: "It was mentioned on CNN that the new prime number discovered recently is four times bigger then the previous record."

Douglas Adams: "There is a theory which states that if ever anybody discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened."

$635318657 = 59^4 + 158^4 = 133^4 + 134^4$ is the smallest number that can be expressed as the sum of two fourth powers in more than one way. (Euler, 1772)

We started in today with our thought questions from the end of last time. How do you build a divisibility test for $n = 11$, or $n = 13$? It was pointed out that for a number with two digits, multiples of 11 definitely follow a pattern! They are just the same number repeated. This sounded like such a great trait that we decided to build a whole test around it. After staring at a large number of multiples of 11,

$$121, 143, 176, 253, 209, 1353, 2574, 13673$$

we could detect other patterns. actually, because your instructor left off numbers like 209, our pattern emerged as: for 3-digit numbers, the 100's and 1's digits add up to the 10's digit (which, OK, 209 fails, but start small...), and for 4-digit numbers the 100's and 10's digits add up to the same number as a 100's and 1's digits. This pattern seemed to carry on for still higher numbers of digits, too.

So is this our pattern? Well, 209 is a multiple of 11, but the pattern doesn't hold. But $2 + 9$ differs from 0 by 11. So maybe the two sums should differ by 0 or 11? Or by a multiple of 11? We can see what's going on if we think about it the way we did for 7: look at what remainders the powers of 10 leave on division by 11:

$10^0 = 1 = 11 \cdot 0 + 1$
$10^1 = 10 = 11 \cdot 0 + 10$
$10^2 = 100 = 11 \cdot 9 + 1$
$10^3 = 1000 = 11 \cdot 90 + 10$
$10^4 = 10000 = 11 \cdot 909 + 1$
$10^5 = 100000 = 11 \cdot 9090 + 10$

As we noted beore, the quotients can be determined from the decimal expansion $1/11 = .090909090909\ldots$ by continually moving the decimal point to the right. Once you 'know' the quotient, you can determine the remainder (which is what we really care about!) by multiplication and subtraction.

The point we discovered, though, is that 100 leaves remainder 1 when divided by 11. So if we write a number N as $100n + r$ (i.e., r is the last two digits and n is the rest), then

$$N = 100n + r = (11 \cdot 9 + 1)n + r = 11(9n) + (n + r)$$

and so n will leave the same remainder on division by 11 as $n + r$ will. Essentially, what has happen is that we have cut N in two, between the 100's and 1000's places, stacked the resulting two numbers on top of one another, added them, and noted that this number has the same remainder on division by 11 as N does. But we can do this again! And again... In the end, we recover exactly what we were observing: if we take our number N, and, from the right, cut it into two digit pieces and add them together, to get Q, then N and Q have the same remainder on division by 11. If our sum Q ends up having more than two digits, aplly this procedure to it, resulting in a still shorter number whose remainder is the same as Q (and so the same as N. This ends with a two digit number, which is a multiple of 11 (precisely when the original N is divisible by 11 when the two digits are the same, which was our original observation!

This same reasoning can be applied to 13 (or, in fact, nearly any other number we want a divisibility test for), the only question is, what size chunks (we decided that was a perfectly fine mathematical term) to carve our number to be tested into. And this involved looking at the powers of 10, modulo 13. To compute our list we cheated a little and consulted calculators and Maple, which agreed that

$$1/13 = .076923076923076923\cdots$$

so we could compute the quotients by taking the integer parts of this after shifting the decimal to the right, and (more or less) compute the remainder by looking at the last digit of the products of 13 and the quotients:

$10^0 = 1 = 13 \cdot 0 + 1$
$10^1 = 10 = 13 \cdot 0 + 10$
$10^2 = 100 = 13 \cdot 7 + 9$
$10^3 = 1000 = 13 \cdot 76 + 12$
$10^4 = 10000 = 13 \cdot 769 + 3$
$10^5 = 100000 = 13 \cdot 7692 + 4$
$10^6 = 1000000 = 13 \cdot 76923 + 1$
$10^7 = 10000000 = 13 \cdot 769230 + 10$
$10^8 = 100000000 = 13 \cdot 7692307 + 9$
$10^9 = 1000000000 = 13 \cdot 76923076 + 12$

In this case, we find that $10^6 \underset{13}{\equiv} 1$, so if we were to write $N = 10^6 n + r$, where here now r represents the last 6 digits of N and n represents the rest of the digits, then N and $n + r$ have the same remainder on division by 13. So just like with 11 there is a particular size of chunk to cut you number N into (reading from the right) - in this case, the size is 6 - so that if we add up the chunks, the resulting sum has the same remainder on division by 13 as your original number has. So the resulting sum is divisble by 13 precisely when the original number is divisible by 13. And we can go further and build a divisiblity test for 6-digit numbers on top of this:

If $N = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + 10^4a_4 + 10^5a_5$, then, by subtracting off the appropriate multiples of 13 (for example, $10^4a_4 = (13 \cdot 769 + 3)a_4 = 13(769a_4) + 3a_4$), we find that N has the same remainder as $a_0 + 10a_1 + 9a_2 + 12a_3 + 3a_4 + 4a_5$ on division by 13. And by swapping large positive numbers for smaller negative numbers (by subtracting $13a_1$, for example), N has the same remainder as $a_0 - 3a_1 - 4a_2 - 1a_3 + 3a_4 + 4a_5$. This gives us a fairly practical divisibility test for 13 (!). And just to not let the opportunity pass by, do you notice any interesting pattern in the coefficients of that last sum?

But we are still not being lazy enough! We are still doing more brute force computation, to figure out the remainders in that chart, than we probably would like to. But if we stop and think a bit about how congruence modulo some number n behaves when we combine it with addition or subtraction or multiplication, we can learn a few more shortcuts! One of your thought quations earlier has a role here: if $17|n-a$ and $17|m-b$, the challenge was to show that $17|na-mb$. In the end we could do this in a couple of ways, one was:

Since $17|n-a$, this means $n-a=17x$, and since $17|m-b$, we have $m-b=17y$. But then $n=17x+a$ and $m=17y+b$, so

$nm-ab=(17x+a)(17y+b)-ab=17x17y+17xb+a17y_ab-ab=17(17xy+xb+ay)$ is a multiple of 17, so $17|nm-ab$. And the real point to that computation is that 17 really had nothing to do with it! Replace 17 with an arbitrary integer d, and we would find that if $d|n-a$ and $d|m-b$ then $d|nm-ab$. Written in a different notation, we have found that

$$\text{If } n \underset{d}{\equiv} m \text{ and } a \underset{d}{\equiv} b \text{, then } nm \underset{d}{\equiv} ab.$$

In still other words, to find the remainder of a product on divsion by d, take the remainders first! Then multiply the remainders together, and take the remainder of <u>that</u> product. The basic idea is that if all we care about is the remainder, we can keep the numbers we are multiplying together small. As we will see, congrueence plays nicely with other basic arithmetic operations, as well! This in turn will help us to carry out manyof the calculations we have been doing (and will do) considerably more efficiently.

To stimulate discussion for Thursday, we finished with the following problem:

If $n \underset{d}{\equiv} m$ and $a \underset{d}{\equiv} b$, show that $n+m \underset{d}{\equiv} a+b$. What about differences, as well?