Joy of Numbers – Take Home Test # 3

Due: Thursday, October 20

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

You will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counterexamples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

1. Show that for any integers a and b and any integer n we have $\gcd(a, b) = \gcd(a, a + b)$. [You can approach this by thinking about the collections of common divisors for each pair, or by studying what the Euclidean algorithm would do for the second pair.]

2. We've already met the Fibonacci sequence F_n, defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ when $n \geq 1$. Show **by induction on** n that the gcd of the consecutive terms F_n and F_{n+1} of the Fibonacci sequence is always 1 (in the terminology we introduced, we are showing that F_n and F_{n+1} are always *relatively prime*. [The previous problem might help!]

3. The Fibonacci sequence F_n grows fairly fast, but not "too" fast, as n gets larger and larger. Show, using (complete!) induction, that for $n \geq 2$ we have $F_n \geq \left(\frac{3}{2}\right)^n$; show <u>also</u> by (complete) induction that for every $n \geq 1$ we have $F_n \leq 2^n$.

   Extra credit: Can you improve on these bounds? Can you find $c > \frac{3}{2}$ and/or $d < 2$ with $F_n > c^n$ and $F_n < d^n$ for every $n \geq 2$? Or for $n \geq n_0$ for some n_0 ?

4. *A divisibility test for* 41. We've built divisibility tests for various numbers n (like 7,11, and 13) by exploring the powers of 10 modulo n, finding the smallest k so that $10^k \underset{n}{\equiv} 1$. What do you discover when you do this for the divisor $n = 41$? What would your divisibility test look like?