Joy of Numbers – Take Home Test # 1

Due: Thursday, September 8

Collaboration Allowed

Collaboration on this test is both allowed and encouraged. By collaboration, I mean that you are encouraged to discuss the problems, test out your ideas, check your reasoning and arguments, etc., with other people. *However, there is a big difference between collaborating and copying. Each student must write up his or her own solutions to the problems in his or her own words.*

You will be graded both on mathematical content and on clarity of expression. Each problem is worth a maximum of 12 points. Some of the questions are a bit open-ended, so be creative, make educated guesses if you have to, but back up your assertions by providing proofs, counterexamples, or (in the case of guesses) numerical evidence. In writing your answers, use complete sentences (with punctuation!) and be sure to say exactly what you mean. Papers will be graded on the basis of what you have written, so be sure to take the time to express yourself clearly. If you are stuck on a problem and have no idea where to begin, a good way to get started is to look at lots of specific examples and try to find a pattern.

1. Every odd integer can be written either as $4n + 1$ or $4n + 3$ for some integer n. Why is this true? We will call these two kinds of numbers $(4, 1)$-numbers and $(4, 3)$-numbers. Do some experiments with multiplying numbers of different kinds together. What can you conclude? Give a formal proof about what happens when two $(4, 3)$-numbers are multiplied together.

2. Explain how to show, using only hand computation (but with the least computation that you can muster), that $N = 1013$ is a prime number.

3. A triple of positive integers (a, b, c) (which we will assume is written in increasing order $a \leq b \leq c$) is called a *Pythagorean triple* if $a^2 + b^2 = c^2$. [You may know that a triangle built with these side lengths will be a right triangle.] The most famous such triple is probably $(3, 4, 5)$. The goal here is to find more! One way to do this is to suppose that c is a fixed amount, say 1 (!), larger than b, so $c = b + 1$, and then ask *"When is $c^2 - b^2 = (b + 1)^2 - b^2$ equal to a^2, i.e., a perfect square?* Use this to find Pythagorean triples with $a = 5, 7, 9, 11$. Describe, if you can, a general procedure to keep going.

4. We will learn that, in some sense, finding factors of a number is a much more challenging problem than knowing that the number is composite! But if the numbers have a particular form, we can use our knowledge about factoring polynomials to find factors of the number. And factoring polynomials can sometimes be done by finding roots of the polynomial.

   By factoring the polynomial $Q(h) = h^2 + 6h + 5$, show that there are very few values of n for which $N = n^2 + 6n + 5$ is prime; what are they?

   The polynomial $R(h) = h^2 + h + 5$ cannot be factored into lower degree polynomials with integer coefficients. Does this mean that $N = n^2 + n + 5$ is always prime? What does some experimentation tell you?