Math 1710
Topics for first exam

Chapter 1: Limits and Continuity

§1: Rates of change and limits

Calculus = Precalculus + (limits)

Limit of a function \( f \) at a point \( x_0 = \) the value the function ‘should’ take at the point
= the value that the points ‘near’ \( x_0 \) tell you \( f \) should have at \( x_0 \)

\[ \lim_{x \to x_0} f(x) = L \] means \( f(x) \) is close to \( L \) when \( x \) is close to (but not \underline{equal} to) \( x_0 \)

Idea: slopes of tangent lines

\[ \lim_{x \to x_0} f(x) = L \] does not care what \( f(x_0) \) is; it ignores it
\[ \lim_{x \to x_0} f(x) \] need not exist! (function can’t make up it’s mind?)

§2: Rules for finding limits

If two functions \( f(x) \) and \( g(x) \) agree (are equal) for every \( x \) near \( a \)
(but maybe not at \( a \)), then \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \)

Ex.: \[ \lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 1}{x + 2} \]

If \( f(x) \to L \) and \( g(x) \to M \) as \( x \to x_0 \) (and \( c \) is a constant), then
\[ f(x) + g(x) \to L + M \; ; \; \; \; f(x) - g(x) \to L - M \; ; \; \; cf(x) \to cL \; ; \; \; f(x)g(x) \to LM \; ; \; \; \text{and} \; \frac{f(x)}{g(x)} \to \frac{L}{M} \; \text{provided} \; M \neq 0 \]

If \( f(x) \) is a polynomial, then \( \lim_{x \to x_0} f(x) = f(x_0) \)

Basic principle: to solve \( \lim_{x \to x_0} \), plug in \( x = x_0 \)!

If (and when) you get \( 0/0 \), try something else! (Factor \( x - x_0 \) out of top and bottom...)

If a function has something like \( \sqrt{x} - \sqrt{a} \) in it, try multiplying (top and bottom) with \( \sqrt{x} + \sqrt{a} \)

Sandwich Theorem: If \( f(x) \leq g(x) \leq h(x) \), for all \( x \) near \( a \) (but not \underline{at} \( a \)), and
\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \quad \text{then} \quad \lim_{x \to a} g(x) = L.
\]

§4: **Extensions of the limit concept**

Motivation: the Heaviside function

![Graph of the Heaviside function](image)

The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, \( H(0) \) 'wants' to be 0, while on the right, \( H(0) \) 'wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the 'one-sided' limits DO exist.

Limit from the right: \( \lim_{x \to a^+} f(x) = L \) means \( f(x) \) is close to \( L \)
when \( x \) is close to, and **bigger than**, \( a \)

Limit from the left: \( \lim_{x \to a^-} f(x) = M \) means \( f(x) \) is close to \( M \)
when \( x \) is close to, and **smaller than**, \( a \)

\[
\lim_{x \to a} f(x) = L, \quad \text{then means} \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L
\]

Infinite limits: \( \infty \) represents something bigger than any number we can think of
\[
\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a} f(x) = -\infty
\]

Also have \( \lim_{x \to a} f(x) \) is really large as \( x \) gets close to \( a \)

\[
\lim_{x \to a^+} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = \infty
\]

Typically, an infinite limit occurs where the denominator of \( f(x) \) is zero
(although not always)

§5: **Continuity**

A function \( f \) is **continuous** (cts) at \( a \) if \( \lim_{x \to a} f(x) = f(a) \)

This means: (1) \( \lim_{x \to a} f(x) \) exists; (2) \( f(a) \) exists; and (3) they're equal.

Limit theorems say (sum, difference, product, quotient) of cts functions are cts.

Polynomials are continuous at every point;

rational functions are continuous except where \( \text{denom}=0 \).

Points where a function is not continuous are called **discontinuities**.

Four flavors:

- **removable**: both one-sided limits are the same
- **jump**: one-sided limits exist, not the same
- **infinite**: one or both one-sided limits is \( \infty \) or \(-\infty \)
Chapter 2: Derivatives

§1: The derivative of a function

Derivative = limit of difference quotient (two flavors)

$f'(x_0)$ exists, say $f$ is differentiable at $x_0$

Fact: $f$ differentiable (differentiable) at $x_0$, then $f$ cts at $x_0$

$h \to 0$ notation: replace $x_0$ with $x$ (= variable), get $f'(x) = \text{new function}$

$f'(x) = \text{the derivative of } f = \text{function whose values are the slopes of the tangent lines to the graph of } y = f(x) \text{. Domain = every point where the limit exists}$

Notation:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{df}{dx} = y' = D_xf = Df = (f(x))'$$

§2: Differentiation rules

$$\frac{d}{dx}(\text{constant}) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$(f(x)+g(x))' = (f(x))' + (g(x))'$$

$$(f(x)-g(x))' = (f(x))' - (g(x))'$$

(c$f(x))' = c(f(x))'$$

$$(f(x)g(x))' = (f(x))'g(x)+f(x)(g(x))'$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

(x^n)' = nx^{n-1}, \text{ for } n=0,1,-1,2,-2,3,......$$

$$((1/g(x))' = -g'g/(g(x))^2)$$

$f'(x)$ is ‘just’ a function, so we can take its derivative!

$$(f'(x))' = f''(x) \quad = y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$$

= second derivative of $f$ (=rate of change of rate of change of $f$ !)

Keep going! $f'''(x) = 3$rd derivative, $f^{(n)}(x) = n$th derivative
§3: Rates of change

Physical interpretation:
\( f(t) = \text{position at time } t \)
\( f'(t) = \text{rate of change of position} = \text{velocity} \)
\( f''(t) = \text{rate of change of velocity} = \text{acceleration} \)
\( |f'(t)| = \text{speed} \)

Basic principle: for object to change direction (velocity changes sign),
\( f'(t) = 0 \) somewhere (IVT!)
Examples:
Free-fall: object falling near earth; \( s(t) = s_0 + v_0 t - \frac{g}{2} t^2 \)
\( s_0 = s(0) = \text{initial position}; v_0 = \text{initial velocity}; g = \text{acceleration due to gravity} \)

Economics:
\( C(x) = \text{cost of making } x \text{ objects}; R(x) = \text{revenue from selling } x \text{ objects}; \)
\( P = R - C = \text{profit} \)
\( C'(x) = \text{marginal cost} = \text{cost of making 'one more' object} \)
\( R'(x) = \text{marginal revenue}; \) profit is maximized when \( P'(x) = 0 \); i.e., \( R'(x) = C'(x) \)

§4: Derivatives of trigonometric functions

Basic limit: \( \lim_{x\to0} \frac{\sin x}{x} = 1 \); everything else comes from this!

Note: this uses radian measure! \( \lim_{x\to0} \frac{\sin(bx)}{x} = b \)

Then we get:
\( (\sin x)' = \cos x \)
\( (\cos x)' = -\sin x \)
\( (\tan x)' = \sec^2 x \)
\( (\cot x)' = -\csc^2 x \)
\( (\sec x)' = \sec x \tan x \)
\( (\csc x)' = -\csc x \cot x \)

§5: The Chain Rule

Composition \( (g \circ f)(x_0) = g(f(x_0)) \); (note: we don’t know what \( g(x_0) \) is.)
\( (g \circ f)' \) ought to have something to do with \( g'(f(x)) \) and \( f'(x) \)

in particular, \( (g \circ f)'(x_0) \) should depend on \( f'(x_0) \) and \( g'(f(x_0)) \)

Chain Rule: \( (g \circ f)'(x_0) = g'(f(x_0))f'(x_0) \)
\( = (d(\text{outside}) \text{ eval'd at inside fcn}) \cdot (d(\text{inside})) \)
Ex: \((x^3 + x - 1)^4)' = (4(x^3 + 1 - 1)^3)(3x^2 + 1) \)

Different notation:
y = g(f(x)) = g(u), where \( u = f(x) \), then \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

§6: Implicit differentiation

We can differentiate functions; what about equations? (e.g., \( x^2 + y^2 = 1 \))

graph looks like it has tangent lines
Idea: Pretend equation defines $y$ as a function of $x$: $x^2 + (f(x))^2 = 1$ and differentiate!

$$2x + 2f(x)f'(x) = 0; \text{ so } f'(x) = \frac{-x}{f(x)} = \frac{-x}{y}$$

Different notation:

$$x^2 + xy^2 - y^3 = 6; \text{ then } 2x + (y^2 + x(2y \frac{dy}{dx}) - 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - y^2}{2xy - 3y^2}$$

Application: extend the power rule

$$\frac{d}{dx}(x^r) = rx^{r-1} \text{ works for any rational number } r$$

§7: Related Rates

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$$xyz = 3; \text{ pretend each is a function of } t, \text{ and differentiate (implicitly).}$$

General procedure:

- Draw a picture, describing the situation; label things with variables.
- Which variables, rates of change do you know, or want to know?
- Find an equation relating the variables whose rates of change you know or want to know.
- Differentiate!
- Plug in the values that you know.