Math 1650

Topics for second exam

(Technically, everything covered on the first exam, plus...)

Chapter 2: Polynomials

§3: Polynomial division

- **root** $a$ of $f \leftrightarrow$ factor $(x - a)$ of $f(x)$
- reason: polynomial (long) division
- $f(x) = (x - a)g(x) + b$; $a$=root, then $b = 0$
- polynomial $= (divisor)(quotient) +$ remainder
  - degree of remainder is less than degree of divisor
- synthetic division: fast method to divide by $(x-a)$

§4: Real zeros of polynomial functions

- $f(x) = a_nx^n + \cdots + a_1x + a_0$
- ‘Counting’ zeros of $f$
- Descartes’ rule of signs
  - $p =$number of positive roots of $f$; $q =$number of negative roots of $f$
  - (number of changes in sign of coeffs of $f$) – $p$ is $\geq 0$ and even
  - (number of changes in sign of coeffs of $f(-x)$) – $q$ is $\geq 0$ and even

Rational roots test

- If $a_n, \ldots, a_0$ are all integers, $a_n \neq 0$, and $r = p/q$ is a rational root of $f$, then
  - $q$ divides $a_n$ evenly and $p$ divides $a_0$ evenly.
  - backwards: can show roots of a polynomial can’t be rational.
- Bounding roots: start with $a_n > 0$.
  - If $c > 0$ and the bottom row after synthetic division of $f$ using $c$ are all $\geq 0$,
    - then no root of $f$ is bigger than $c$.
  - If $c < 0$ and the bottom alternates sign, then no root of $f$ is smaller than $c$.

§5: Complex numbers

- Some polynomials have no roots, e.g., $f(x) = x^2 + 1$. Invent some!
  - $i = \sqrt{-1}$, pretend $i$ behaves like a real number
  - complex numbers: standard form $z = a + bi$; addition, subtraction, multiplication
  - division: complex conjugate $\overline{z} = a - bi$
  - $z \cdot \overline{z} = a^2 + b^2$ (real!); $z_1/z_2 = (z_1 \cdot \overline{z_2})/(z_2 \cdot \overline{z_2})$
  - $a, b > 0$, then $\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{(-a)(-b)}$ (unfortunately)

§6: The fundamental theorem of algebra

- FTA: Every polynomial $f(x)$ (with coefficients in $\mathbb{C}$ or $\mathbb{R}$) has a complex root $r$; $f(r)=0$
- Every polynomial factors into linear factors (with coefficients in $\mathbb{C}$)
- FTA says it can be done; it doesn’t tell you how to do it!
- Conjugate pairs: if coeffs of $f$ are real and $r$ is a root, then so is $\overline{r}$
  - $(x - r)(x - \overline{r})$ has real coeffs
- every polynomial with real coeffs factors in linear and irreducible quadratic factors.

§7: Rational functions

- rational function $= $ quotient of polynomials
  - $p(x) = a_nx^n + \cdots + a_0$; $q(x) = b_mx^m + \cdots + b_0$; $f(x) = p(x)/q(x)$
  - domain $= \text{ where } q(x) \neq 0$
- vertical asymptote $x = a$; $f(x) \to \pm \infty$ as $x \to a$
- horizontal asymptote: $f(x) \to a$ as $x \to \pm \infty$
  - $n < m$: horiz. asymp. $y = 0$
  - $n = m$: horiz. asymp. $y = a_n/b_m$
Chapter 3: Exponential and logarithmic functions

§1: Exponential functions

exponential expressions $a^b$
- Rules: $a^{b+c} = a^b a^c$ ; $a^{bc} = (a^b)^c$ ; $(ab)^c = a^c b^c$
- Function $f(x) = a^x$ ; approximate $f(x)$ by $f(\text{rational number close to } x)$
  - Domain: $\mathbb{R}$ ; range: $(0, \infty)$ ; horiz. asymp. $y = 0$
  - Graphs:

  $a > 1$
  - Most natural base: $e = 2.718281829459045$.....
  - Exponential growth: compound interest
    - $P=$initial amount, $r=$interest rate, compounded $n$ times/year
    - $A(t) = P \cdot (1 + r/n)^{nt}$
    - $n \to \infty$, continuous compounding : $A(t) = Pe^{rt}$
  - Radioactive decay: half-life = $k$ ($A(k) = A(0)/2$)
    - $A(t) = A(0)(1/2)^{t/k}$

§2: Logarithmic functions

log $x$ = the number you raise $a$ to to get $x$
- log $x$ is the inverse of $a^x$
- $a =$ base of the logarithm
- log $a(x^x) = x$, all $x$ ; $a^{\log_a x} = x$, all $x > 0$
- Domain: all $x > 0$ ; range: all $x$
- Graph = reflection of graph of $a^x$ across line $y = x$
  - vertical asymptote: $x = 0$
- natural logarithm: log $e$, $x = \ln x$

§3: Properties of logarithms

logarithms undo exponentials; properties are ‘reverse’ of exponentials
- $\log_a (bc) = \log_a b + \log_a c$ ; $\log_a (b^c) = c \log_a b$
- $(\log_b c)(\log_a b) = \log_a (b^{\log_b c}) = \log_a c$ ; so $\log_b c = \frac{\log_a c}{\log_a b}$
- E.g., $a = e$ : $\log_b c = \frac{\ln c}{\ln b}$

§4: Exponential and logarithmic equations

exponential equation: take logs!
- $a^{\text{blah}} = \text{bleh}$, then $(\text{blah}) \ln a = \ln(\text{bleh})$
- $(2^x - 3)(2^x - 7) = 0$, then $2^x = 3$ or $2^x = 7$

logarithmic equation: combine into a single log (one on each side?) and exponentiate both sides
Application: doubling time
time for investment to triple at interest rate of $r$ compounded $n$ times/year:
solve $(1 + r/n)^{nt} = 3$