Math 106 Exam 3 Topics

Newton's method:

A really fast way to approximate roots of a function.
Idea: tangent line to the graph of a function "points towards" a root of the function. But: roots of (tangent) lines are computationally straighforward to find!

$L(x) = f(x_0) + f'(x_0)(x - x_0)$; root is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Now use x_1 as starting point for new tangent line; keep repeating!

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Basic fact: if x_n approximates a root to k decimal places, then x_{n+1} tends to approximate it to $2k$ decimal places! BUT:

Newton's method might find the "wrong" root: Int Value Thm might find one, but N.M. finds a different one!
Newton's method might crash: if $f'(x_n) = 0$, then we can't find x_{n+1} (horizontal lines don't have roots!)
Newton's method might wander off to infinity, if f has a horizontal asymptote; an initial guess too far out the line will generate numbers even farther out.
Newton's method can't find what doesn't exist! If f has no roots, Newton's method will try to "find" the function's closest approach to the x-axis; but everytime it gets close, a nearly horizontal tangent line sends it zooming off again...

Optimization

This is really just finding the max or min of a function on an interval, with the added complication that you need to figure out *which* function, and *which* interval! Solution strategy is similar to a *related rates* problem:

Draw a picture; label things.
What do you need to maximize/minimize? Write down a formula for the quantity.
Use other information to eliminate variables, so your quantity depends on only one variable.
Determine the largest/smallest that the variable can reasonably be (i.e., find your interval)
Turn on the max/min machine!

L'Hôpital's Rule

indeterminate forms: limits which 'evaluate' to 0/0 ; e.g. $\lim_{x\to 0} \dfrac{\sin x}{x}$

LR# 1: If $f(a) = g(a) = 0$, f and g both differentiable near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note: we can repeatedly apply L'Hôpital's rule to compute a limit, so long as the condition that top and bottom both tend to 0 holds for the new limit. Once this doesn't hold, L'Hôpital's rule can no longer be applied!

Other indeterminate forms: $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

LR#2: if $f, g \to \infty$ as $x \to a$, then $\quad \displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Other cases: try to turn them into $0/0$ or ∞/∞. In the $0 \cdot \infty$ case, we can do this by throwing one factor or the other into the denomenator (whichever is more tractable. In the last three cases, do this by taking logs, first.

Antiderivatives.

Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.
Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it <u>means</u> $F'(x) = f(x)$; "the integral of f of x dee x"

Every differentiation formula we have encountered can be turned into an antidifferentiation formula; if g is the derivative of f, then f is an antiderivative of g. Two functions with the same derivative (on an interval) differ by a constant, so <u>all</u> antiderivatives of a function can be found by finding one of them, and then adding an arbitrary constant C.

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int e^x\ \mathrm{d}x = e^x + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some we will wait awhile to discover).

Basic integration rules: sum and constant multiple rules are straighforward to reverse: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

Sums and Sigma Notation.

Idea: a lot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

Formal properties: $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Some things worth adding up:
length of a curve: approximate curve by a collection of straight line segments

length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)
over short periods of time, avg. vel. $\approx$ instantaneous vel.
so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

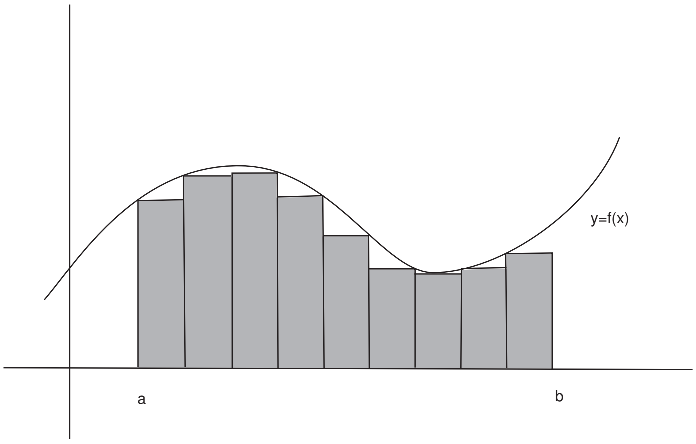
Average value of a function:

Average of n numbers: add the numbers, divide by n . For a function, add up lots of values of f, divide by number of values.

avg. value of $f \approx \frac{1}{n}\sum_{i=1}^{n} f(c_i)$

Area and Definite Integrals.

Probably the most important thing to approximate by sums: area under a curve.
Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate: **rectangles**!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $=\sum_{i=1}^{n} f(c_i)\Delta x_i$

where c_i is chosen inside of the i-th interval that we cut $[a, b]$ up into. This is a Riemann sum for the function f on the interval $[a, b]$.)

We define the area to be the limit of these sums as the lengths of the subintervals gets small (so the number of rectangles goes to ∞, and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\ dx \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

More precisely, we can at all Riemann sums, and look at what happens when the length Δx_i of the largest subinterval (call it Δ) gets small. If the Riemann sums all approximate some number I when Δ is small enough, then we call I the definite integral of f from a to b. But when do such limits exist?

Theorem If f is continuous on the interval $[a, b]$, then $\displaystyle\int_a^b f(x)\ dx$ exists.

(i.e., the area under the graph is approximated by rectangles.)

But this isn't how we want to compute these integrals! Limits of sums is very cumbersome. Instead, we try to be more systematic.

Properties of definite integrals:

First note: the sum used to define a definite integral doesn't <u>need</u> to have $f(x) \geq 0$; the limit still makes sense. When f <u>is</u> bigger than 0, we <u>interpret</u> the integral as area under the graph.

Basic properties of definite integrals:

$$\int_a^a f(x)\ dx = 0 \qquad \int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$$

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx \qquad \int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$

$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$

More generally, if $f(x) \leq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

Average value of f : formalize our old idea! $\quad \text{avg}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$

Mean Value Theorem for integrals: If f is continuous in $[a,b]$, then there is a c in $[a,b]$ so that $f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$

The fundamental theorems of calculus.

Formally, $\int_a^b f(x)\ dx$ <u>depends</u> on a and b. Make this explicit:

$\int_a^x f(t)\ dt = F(x)$ is a function of x.

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 1): If f is continuous, then $F'(x) = f(x)$ (F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t)\ dt$, we get

Fund. Thm. of Calc (# 2): If f is continuous, and F is an antiderivative of f, then $\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\ |_a^b$

Ex: $\int_0^\pi \sin x\ dx = (-\cos\pi) - (-\cos 0) = 2$

FTC # 2 makes <u>finding</u> antiderivatives very important! FTC # 1 gives a method for building antiderivatives:

$F(x) = \int_a^x \sqrt{\sin t}\ dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \int_{x^2}^{x^3} \sqrt{1+t^2}\ dt = F(x^3) - F(x^2)$, where
$F'(x) = \sqrt{1+x^2}$, so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x)$...

Integration by substitution.

The rules we have tell us that the sums, differences, and constant multiples of functions whose integrals we can handle we can <u>also</u> handle. Further rules allow us to relate the antiderivatives of functions to the antiderivatives of "simpler" functions.

The idea: reverse the chain rule!

If $g(x) = u$, then $\frac{d}{dx} f(g(x)) = \frac{d}{dx} f(u) = f'(u)\ \frac{du}{dx}$

so $\int f'(u)\ \frac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

So: faced with $\int f(g(x))g'(x)\ dx$, set $u = g(x)$, then $du = g'(x)\ dx$, so $\int f(g(x))g'(x)\ dx = \int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x+2-3)^4\ dx$; set $u = x^2 - 3$, so $du{=}2x\ dx$. Then

$\int x(x+2-3)^4\ dx = \frac{1}{2}\int (x+2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} =$
$\frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$

$\int \tan x\ dx = \int \frac{\sec x \tan x}{\sec x}\ dx$; using the substitution $u = \sec x$ we get $\int \tan x\ dx = \ln|\sec x| + C = -\ln|\cos x| + C$. Similarly, $\int \csc x\ dx = \ln|\sin x| + c$.

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. <u>**Don't**</u> push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that $\int_a^b f(x)\ dx$ <u>really</u> means $\int_{x=a}^{x=b} f(x)\ dx$, and we remember to change <u>all</u> of the x's to u's!

Ex: $\int_1^2 x(1+x^2)^6\ dx$; set $u = 1+x^2$, $du = 2x\ dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$; so $\int_1^2 x(1+x^2)^6\ dx = \frac{1}{2}\int_2^5 u^6\ du = ...$