Math 107H
Topics for the second exam

Technically, everything for the first exam! Plus:

Improper integrals

Fund Thm of Calc: $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, where $F'(x) = f(x)$

Problems: $a = -\infty$, $b = \infty$; f blows up at a or b or somewhere in between
integral is "improper"; usual technique doesn't work. Solution to this:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x \qquad \int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x$$

(blow up at a) $\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to a^+} \int_r^b f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)\,\mathrm{d}x$

(similarly for blowup at b (or both!))

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s\to b^-} \int_a^s f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

(blows up at c (b/w a and b)) $\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to c^-} \int_a^r f(x)\,\mathrm{d}x + \lim_{s\to c^+} \int_s^b f(x)\,\mathrm{d}x$

The integral converges if (all of the) limit(s) are finite

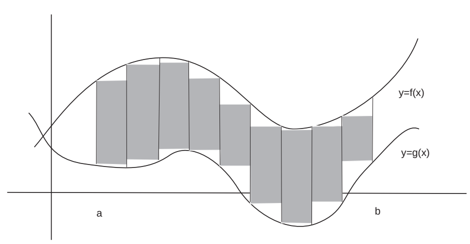
Comparison: $0 \le f(x) \le g(x)$ for all x;

if $\int_a^\infty g(x)\,\mathrm{d}x$ *converges*, so does $\int_a^\infty f(x)\,\mathrm{d}x$;

if $\int_a^\infty f(x)\,\mathrm{d}x$ *diverges*, so does $\int_a^\infty g(x)\,\mathrm{d}x$.

Applications of integration

Area between curves. Region between two curves; approximate by rectangles



$$\text{Area} = \int_{left}^{right} (top) - (bottom)\; dx = \int_a^b f(x) - g(x)\; dx$$

Integrate dy: Area $= \int_{bottom}^{top} (right) - (left)\; dy$

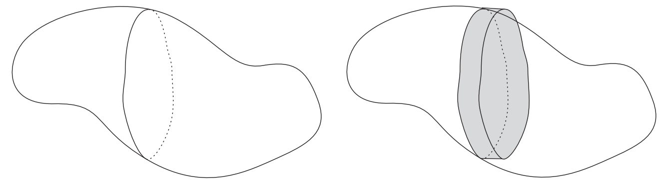
Which function is top/bottom changes? Cut interval into pieces, and use $\int_a^b = \int_a^c + \int_c^b$

Sometimes to calculate area between $f(x)$ and $g(x)$, need to first figure out limits of integration; solve $f(x) = g(x)$, then decide which one is bigger in between each pair of solutions.

Volume by slicing. To calculate volume, aprroximate region by objects whose volume we can calculate.

Volume $\approx \sum$(volumes of 'cylinders')
$= \sum$(area of base)(height)
$= \sum$(area of cross-section)Δx_i .

So volume $= \displaystyle\int_{left}^{right}$ (area of cross section) dx

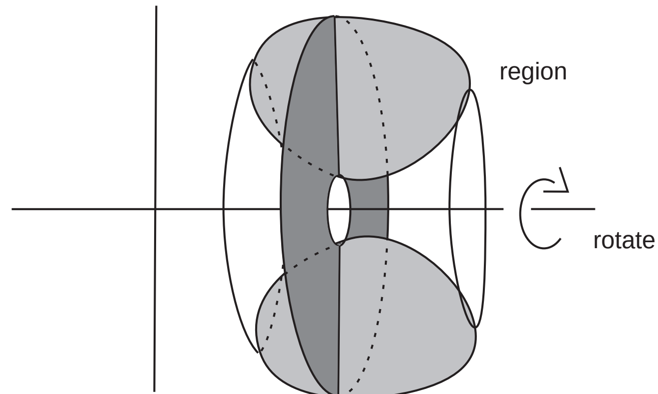


Solids of revolution: disks and washers. Solid of revolution: take a region in the plane and revolve it around an axis in the plane.

take cross-sections perpendicular to axis of revolution;
cross-section = disk (area$=\pi r^2$)
or washer (area$=\pi R^2 - \pi r^2$)

rotate around x-axis: write r (and R) as functions of x, integrate dx

rotate around y-axis: write r (and R) as functions of y, integrate dy



Otherwise, everything is as before: volume $= \displaystyle\int_{left}^{right} A(x)\, dx$ or volume $= \displaystyle\int_{bottom}^{top} A(y)\, dy$

The same is true if axis is parallel to $x-$ or $y-$axis; r and R just change (we add a constant).

Volume by cylindrical shells.

A different approach to solids of revolution: use lines *parallel* to the axis we revolve around. The segment meeting the region R in the plane, when spun, is a cylinder, with area (circumference)$\times$(height). These cylinders sweep out the solid of revolution, as the segments sweep out the region R, and the volume of the solid is the integral of the areas of these cylinders (since a thickened cylinder has volume $\approx$(circumference)(height)Δx. But! Circumference and height can be computed from knowing the region and the axis; if we spin around a vertical line $x = k$, then circumference is $2\pi|x-k|$ and height is "top $-$ bottom", and so if vertical lines hitting R run from $x = a$ to $x = b$, then

volume spun around $x = k$ is $\int_a^b 2\pi|x-k|$(top-bottom) dx.

If we spin R around a horizontal line $y = k$ instead, then cylindrical shells has us meet R in horizontal lines, which meet R in segments from 'left' to 'right'; the radius of our cylindrical shell will be $|y-k|$ and the volume when spun around $y = k$ is $\int_{bottom}^{top} 2\pi|y-k|$(right-left) dy.

For example, if we spin the region under $y = \sin x$ between $x = 0$ and $x = \pi$ around the line $x = -2$, then the volume of the solid of revolution can be computed as $\int_0^\pi 2\pi|x-(-2)|(\sin x - 0)\, dx = \int_0^\pi 2\pi(x+2)\sin x\, dx$.

Arclength. Idea: approximate a curve by lots of short line segments; length of curve $\approx$ sum of lengths of line segments.

A parametric curve is the path traced out by a point moving in the plane. To describe its position at time t, we need to know its coordinates: $x = x(t), y = y(t)$. Line segment between $(x(t_i), y(t_i))$ and $(x(t_{i+1}), y(t_{i+1}))$ has length

$$\sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2}$$

$$= \sqrt{[\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}]^2 + [\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}]^2} \cdot (t_{i+1} - t_i) \approx \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2} \cdot \Delta t_i$$

$$\underline{\text{So}} \text{ length of curve} = \int_{\text{start}}^{\text{stop}} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

The problem: integrating $\sqrt{[x'(t)]^2+[y'(t)]^2}$! Sometimes, $[x'(t)]^2+[y'(t)]^2$ turns out to be a perfect square.....

Special case: curve is the graph of a function $y=f(x)$. Parametrize: $x=t$, $y=f(t)$, so

$$\text{length of curve} = \int_{\text{left}}^{\text{right}} \sqrt{1+[f'(t)]^2}\ dt = \int_a^b \sqrt{1+[f'(x)]^2}\ dx$$

Surface area. When we spin a curve around an axis, we create a *surface of revolution.* We can compute the area of this surface by borrowing ideas from arclength: approximate the curve by line segments, and spin them around the axis. When you do this (if the segments don't meet the axis) you get a piece of the surface of a cone, known as a *frustum* of a cone. By cutting the frustum along the segment we spun and laying it flat in the plane, we get a piece of a washer (because the circumference of the washer is bigger than the circumference of the cone. A more or less routine computation shows that if you spin the line segment between $(a,f(a)$ and $(b,f(b))$ around the x-axis (to keep things concrete), the resulting frustum as area

$$2\pi\frac{f(a)+f(b)}{2}\sqrt{(b-a)^2+(f(b)-f(a))^2}\ .$$

So if we cut the interval from a to b (which is the piece of the graph $y=f(x)$ we will spin) into pieces, and build the frusta for each line segment we use to approximate the graph, and add them together, we find that the surface area is approximated by

$$\sum 2\pi f(x_i)\sqrt{\Delta x_i^2+(f(x_{i+1})-f(x_i))^2} = \sum 2\pi f(x_i)\sqrt{1+(\tfrac{f(x_{i+1})-f(x_i)}{\Delta x_i})2}\cdot\Delta x_i$$
$$\approx 2\pi f(x_i)\sqrt{1+[f'(x_i)]2}\cdot\Delta x_i\ ,$$

and so the surface area of a surface of revolution, spinning $y=f(x)$ between $x=a$ and $x=b$ around the x-axis, is

$\text{SA} = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]2}\ dx.$

(An analogous formula can be established for spinning around the y-axis, or around axes parallel to these ($x=k$, etc.).)

For example, spinning $y=\sin x$ from $x=0$ to $x=\pi$ around the x-axis yields a surface of revolution whose surface area is $\int_0^\pi 2\pi\sin x\sqrt{1+\cos^2 x}\ dx$, an integral we can compute using the substitution $u=\cos x$, followed by the trig substitution $u=\tan v$ ....

Work. In physics, one studies the behavior of objects when acted upon by various *forces.* Newton's Laws provide the basic connection between a force acting on an object and the effect it has on its motion:

$$F=ma\ ; \qquad \text{Force} = \text{mass}\times\text{acceleration}$$

In physics, *work* represents force being applied across a distance. If a constant force F is applied to an object, which moves the object a distance D, then the work done on the object is $W=F\cdot D$. Again, if the force applied across this distance is not constant, then we interpret work, in stead, as an integral, by cutting the distance covered into small pieces of length δx :

$$W\approx\sum F(x_i)\ \Delta x\ \text{, so } W=\int_0^D F(x)\ dx$$

An interesting application of these ideas comes when trying to compute the amount of work necessary to pump out a tank of some known shape. If the tank has height D (we will think of the top of the tank as being at $x=0$ and the bottom being at $x=D$), and at height X our cross-section of the tank has area $A(x)$, then if (as when we computed volume) we think of the fluid in the tank as being a stack of cylinders with height Δx, the work necessary to lift the slice at height x to the top of the tank will be

$$W=\text{(force)(distance)} = (m\cdot g)\cdot x = ((A(x)\cdot\Delta x)\rho g)\cdot x$$

where ρ is the density of the fluid, m = mass = (volume)(density), and g is the accelration due to gravity (which is the force we need to overcome to push the fluid up out of the tank). Therefore, the work done to empty the tank is approximated by a sum of such quantities, which in turn models a definite integral; the work done in emptying the tank is

$$W = \rho g \int_0^D xA(x)\ dx$$

Centers of mass/centroids

The Principle of the Lever tells us that a small mass far from a blanace point can offset a larger mass closer to the balance point. More specifically, a collection $\{m_i\}$ of masses placed at points $\{x_i\}$ along a line will balance at the point $\overline{x}$ where $\sum m_i\overline{x} = \sum m_i x_i$. If an object's mass is more 'diffuse', this passes, in a limit, to an integral: $\overline{x} \in_a^b m(x)\ dx = \int_a^b xm(x)\ dx$, where $m(x)$ is a 'density' function.

Passing to two dimensions, we can determine the point where a region R in the plane (with uniform density, for the sake of simplicity) will balance, by equating 'density' with 'the lengths of parallel line segments', to compute the vertical and horizontal lines that the region R will balance along. If we denote by $h(x)$ the length of the vertical line segment at x meeting R, and by $w(y)$ the length of the horizontal line segment at y meeting R, then we have

$$\overline{x}\int_a^b h(x)\ dx = \int_a^b xh(x)\ dx \qquad \text{and} \qquad \overline{y}\int_c^d w(y)\ dy = \int_c^d yw(y)\ dy,$$

where the region R sits inside of the box $a \le x \le b$ and $c \le y \le d$. Noting, however, then $\int_a^b h(x)\ dx = \int_c^d w(y)\ dy =$ the area of $R = A(R)$, we can replace these with a single computation (or rely on geometry to compute the area). This allows us to compute th *centroid* of R, the point at which R would balance on the point of a pin, as

$$(\overline{x},\overline{y}) = (\frac{1}{A(R)}\int_a^b xh(x)\ dx, \frac{1}{A(R)}\int_c^d yw(y)\ dy).$$

These computations are related to our volume computations: cylindrical shells computations can be interpreted as saying that volume is equal to $A(R)\times$(the distance travelled by the centroid around the axis of revolution). This result is known as the *Theorem of Pappus*. If the centroid can be determined by 'other means' (e.g., the centroid lies on any axis of symmetry of the region R), this can greatly speed up volume computations!

Infinite sequences and series

Limits of sequences of numbers

A sequence is: a string of numbers; a function f:$\mathbf{N}\to\mathbf{R}$; write $f(n) = a_n$

$a_n = n$-th term of the sequence

Basic question: convergence/divergence: $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$) if

eventually all of the a_n are always as close to L as we like, i.e.

for any $\epsilon > 0$, there is an N so that if $n \ge N$ then $|a_n - L| < \epsilon$

Ex.: $a_n = 1/n$ converges to 0 ; can always choose N=$1/\epsilon$

$a_n = (-1)^n$ diverges; terms of the sequence never settle down to a <u>single</u> number

If $a_n = f(n)$ for $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x\to\infty} f(x) = L$, then $a_n \to L$ as $n \to \infty$

(allows us to use L'Hôpital's Rule!)

If a_n is increasing ($a_{n+1} \ge a_n$ for every n) and bounded from above

($a_n \le M$ for every n, for some M) , then a_n converges (but not necessarily to M !)

limit is smallest number bigger than all of the terms of the sequence

Heirarchy of 'blowing up': $\ln(n) << n << n^2 << n^k << 2^n << e^n << n! << n^n$

Limit theorems for sequences

Idea: limits of sequences are a lot like limits of functions

If $a_n \to L$ and $b_n \to M$, then

$(a_n + b_n \to L + M \qquad (a_n - b_n) \to L - M \qquad (a_n b_n) \to LM$, and

$(a_n/b_n) \to L/M$ (provided M, all b_n are $\neq 0$)

Squeeze play theorem: if $a_n \leq b_n \leq c_n$ (for all n large enough) and $a_n \to L$ and $c_n \to L$, then $b_n \to L$

If $a_n \to L$ and f:**R**→**R** is continuous at L, then $f(a_n) \to f(L)$

Another basic list: (x = fixed number, k = konstant)

$$\frac{1}{n} \to 0 \qquad k \to k \qquad x^{\frac{1}{n}} \to 1$$

$$n^{\frac{1}{n}} \to 1 \qquad (1+\frac{x}{n})^n \to e^x \qquad \frac{x^n}{n!} \to 0$$

$x^n \to \{$ 0, if $|x| < 1$; 1, if $x = 1$; diverges, otherwise $\}$

Infinite series

An infinite series is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n \qquad \text{(summation notation)}$$

n-th term of series = a_n ; N-th partial sum of series = $s_N = \sum_{n=1}^{N} a_n$

An infinite series **converges** if the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$ converges

We may start the series anywhere: $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=3437}^{\infty} a_n$, etc. ;

convergence is unaffected (but the number it adds up to is!)

Ex. geometric series: $a_n = ar^n$; $\sum_{k=0}^{n} a_k = a\frac{r^{n+1}-1}{r-1}$; $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$

if $|r| < 1$; otherwise, the series **diverges**.

Application: compound interest. Principal P earning interest rate r each time period, then amount accumulated after n time periods is

$$(1+r)^n P = (1+r)(1+r)^{n-1}P = (1+r)^{n-1}P + r(1+r)^{n-1}P$$

= (amount in account at time $n-1$) + (interest earned in n-th time interval).

If P is deposited each time period, then amount after n is

$$P\frac{(1+r)^{n+1}-1}{(1+r)-1} = \sum_{k=0}^{n}(1+r)^k P = P + (1+r)\sum_{k=0}^{n-1}(1+r)^k P$$

= (deposit at time n) + (amount in account at time $n-1$)

+ (interest earned on amount present at time $n-1$)

Ex. Telescoping series: partial sums s_N 'collapse' to a compact expression

E.g. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})$; $s_N = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - (\frac{1}{N+1} + \frac{1}{N+2}))$

n-th term test: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

So if the n-th terms **don't** go to 0, then $\sum_{n=1}^{\infty} a_n$ diverges

Basic limit theorems: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \qquad \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty} a_n$$

Truncating a series: $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=N}^{\infty} a_n + \sum_{n=1}^{N-1} a_n$

The integral test

Idea: $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n, then the partial sums $\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\, \mathrm{d}x \leq s_N = \sum_{n=a}^{N} a_n \leq \int_a^N f(x)\, \mathrm{d}x$$

so $\displaystyle\sum_{n=a}^{\infty} a_n$ converges exactly when $\displaystyle\int_a^{\infty} f(x)\, \mathrm{d}x$ converges

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ (p-series)

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges exactly when $p > 1$ (logarithmic p-series?)

These families of series make good test cases for comparison with more involved terms (see below!)

Comparison tests

Again, think $\displaystyle\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$; if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\displaystyle\frac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ and $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1))$ diverges; L-comp with $\sum 1/(n \ln n)$