EXERCISES 2.2

1. For the function \( g(x) \) graphed here, find the following limits or explain why they do not exist.
   a. \( \lim_{x \to 1} g(x) \)
   b. \( \lim_{x \to 2} g(x) \)
   c. \( \lim_{x \to 3} g(x) \)

   \[
   y = g(x)
   \]

   \[
   1 \quad 2 \quad 3 \quad x
   \]

   \[
   y = g(x)
   \]

   \[
   1 \quad 2 \quad 3 \quad x
   \]

2. For the function \( f(t) \) graphed here, find the following limits or explain why they do not exist.
   a. \( \lim_{t \to 2} f(t) \)
   b. \( \lim_{t \to 1} f(t) \)
   c. \( \lim_{t \to 0} f(t) \)

   \[
   s = f(t)
   \]

   \[
   -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad t
   \]

   \[
   s = f(t)
   \]

   \[
   -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad t
   \]

3. Which of the following statements about the function \( y = f(x) \) graphed here are true, and which are false?
   a. \( \lim_{x \to 0} f(x) \) exists.
   b. \( \lim_{x \to 0} f(x) = 0. \)
   c. \( \lim_{x \to 0} f(x) = 1. \)
   d. \( \lim_{x \to 1} f(x) = 0. \)
   e. \( \lim_{x \to 1} f(x) = 0. \)
   f. \( \lim_{x \to x_0} f(x) \) exists at every point \( x_0 \) in \((-1, 1).\)

4. Which of the following statements about the function \( y = f(x) \) graphed here are true, and which are false?
   a. \( \lim_{x \to 2} f(x) \) does not exist.
   b. \( \lim_{x \to 2} f(x) = 2. \)
   c. \( \lim_{x \to 1} f(x) \) does not exist.
   d. \( \lim_{x \to x_0} f(x) \) exists at every point \( x_0 \) in \((-1, 1).\)
   e. \( \lim_{x \to x_0} f(x) \) exists at every point \( x_0 \) in \((1, 3).\)

5. \( \lim_{x \to 0} \frac{x}{x-1} \)

6. \( \lim_{x \to 0} \frac{1}{x} \)

7. Suppose that a function \( f(x) \) is defined for all real values of \( x \) except \( x = x_0. \) Can anything be said about the existence of \( \lim_{x \to x_0} f(x) ? \) Give reasons for your answer.

8. Suppose that a function \( f(x) \) is defined for all \( x \) in \([-1, 1]. \) Can anything be said about the existence of \( \lim_{x \to x_0} f(x) ? \) Give reasons for your answer.

9. If \( \lim_{x \to 1} f(x) = 5, \) must \( f \) be defined at \( x = 1? \) If it is, must \( f(1) = 5? \) Can we conclude anything about the values of \( f \) at \( x = 1? \) Explain.

10. If \( f(1) = 5, \) must \( \lim_{x \to 1} f(x) \) exist? If it does, then must \( \lim_{x \to 1} f(x) = 5? \) Can we conclude anything about \( \lim_{x \to 1} f(x)? \) Explain.

Find the limits in Exercises 11–28.

11. \( \lim_{x \to -7} (2x + 5) \)

12. \( \lim_{x \to 12} (10 - 3x) \)

13. \( \lim_{x \to 2} (-x^2 + 5x - 2) \)

14. \( \lim_{x \to -2} (x^3 - 2x^2 + 4x + 8) \)

15. \( \lim_{t \to 6} 8(t - 5)(t - 7) \)

16. \( \lim_{x \to 2/3} 3x(2x - 1) \)

17. \( \lim_{x \to 2} x + 3 \)

18. \( \lim_{x \to 5} x - 7 \)

19. \( \lim_{y \to -5} \frac{y^2}{5 - y} \)

20. \( \lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6} \)

21. \( \lim_{x \to 1} 3(2x - 1)^3 \)

22. \( \lim_{x \to -4} (x + 3)^{1/3} \)

23. \( \lim_{y \to 3} (5 - y)^{2/3} \)

24. \( \lim_{y \to 0} (2y - 8)^{1/3} \)

25. \( \lim_{h \to 0} \frac{\sqrt{3h + 1} + 1}{h} \)

26. \( \lim_{h \to 0} \frac{\sqrt{5h + 4} + 2}{h} \)

27. \( \lim_{h \to 0} \frac{\sqrt{3h + 1} - 1}{h} \)

28. \( \lim_{h \to 0} \frac{\sqrt{5h + 4} - 2}{h} \)
Find the limits in Exercises 29–46.

29. \( \lim_{x \to -3} \frac{x - 5}{x^2 - 25} \)
30. \( \lim_{x \to 3} \frac{x + 3}{x^2 + 4x + 3} \)
31. \( \lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5} \)
32. \( \lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2} \)
33. \( \lim_{l \to 1} \frac{t^2 + t - 2}{t^2 - 1} \)
34. \( \lim_{l \to 1} \frac{t^2 + 3t + 2}{t^2 - t - 2} \)
35. \( \lim_{x \to -2} \frac{-2x - 4}{x^2 + 2x^2} \)
36. \( \lim_{x \to 0} \frac{3y^2 - 16y^2}{9y^3} \)
37. \( \lim_{v \to -1} \frac{v^4 - 1}{v^3 - 1} \)
38. \( \lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16} \)
39. \( \lim_{x \to 3} \frac{\sqrt{x} - 3}{x - 9} \)
40. \( \lim_{x \to -4} \frac{4x - x^2}{2 - \sqrt{x}} \)
41. \( \lim_{x \to 1} \frac{x - 1}{\sqrt{x} + 3 - 2} \)
42. \( \lim_{x \to -1} \frac{\sqrt{x^2 + 8 - 3}}{x + 1} \)
43. \( \lim_{x \to 2} \frac{\sqrt{x^2 + 12 - 4}}{x - 2} \)
44. \( \lim_{x \to -2} \frac{x + 2}{\sqrt{x^2 + 5 - 3}} \)
45. \( \lim_{x \to 3} \frac{2 - \sqrt{x^2 - 5}}{x - 3} \)
46. \( \lim_{x \to -4} \frac{4 - x}{\sqrt{x^2 + 9}} \)

Find the limits in Exercises 47–54.

47. \( \lim_{x \to 0} (2 \sin x - 1) \)
48. \( \lim_{x \to 0} \sin^2 x \)
49. \( \lim_{x \to \infty} x \tan x \)
50. \( \lim_{x \to 0} \frac{1 + x + \sin x}{3 \cos x} \)
51. \( \lim_{x \to 0} \frac{\sqrt{x} + 1 \cos \sqrt{x}}{\cos x} \)
52. \( \lim_{x \to 0} \frac{(x^2 - 1)(2 - \cos x)}{x^4} \)

55. Suppose \( \lim_{x \to 0} f(x) = 1 \) and \( \lim_{x \to 0} g(x) = -5 \). Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

\[
\lim_{x \to 0} \frac{2f(x) - g(x)}{(f(x) + g(x))^{2/3}} = \frac{\lim_{x \to 0} 2f(x) - g(x)}{\left( \lim_{x \to 0} (f(x) + g(x)) \right)^{2/3}}
\]

56. Let \( \lim_{x \to 1} h(x) = 5 \), \( \lim_{x \to -1} p(x) = 1 \), and \( \lim_{x \to 2} r(x) = 2 \). Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

\[
\lim_{x \to 1} \frac{\sqrt{5}h(x)}{p(x)(4 - r(x))} = \frac{\lim_{x \to 1} \sqrt{5}h(x)}{\lim_{x \to 1} p(x)(4 - r(x))}
\]
7 b. Graph $y = 1 - (x^2/6)$, $y = (\sin x)/(2 - 2 \cos x)$, and $y = 1$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \to 0$.

70. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < 0$$

hold for values of $x$ close to zero. (They do, as you will see in Section 8.9.) What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

Give reasons for your answer.

7 b. Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \to 0$.

You will find a graphing calculator useful for Exercises 71–80.

71. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of $f$ at the points $x = -3.1$, $-3.01$, $-3.001$, and so on as far as your calculator can go.

Then estimate $\lim_{x \to -3} f(x)$. What estimate do you arrive at if you evaluate $f$ at $x = -2.9$, $-2.99$, $-2.999$, ... instead?

b. Support your conclusions in part (a) by graphing $f$ near $x_0 = -3$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -3$.

c. Find $\lim_{x \to -3} f(x)$ algebraically, as in Example 7.

72. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of $g$ at the points $x = 1.4$, $1.41$, $1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \to \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing $g$ near $x_0 = \sqrt{2}$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x \to \sqrt{2}} g(x)$ algebraically.

73. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of $G$ at $x = -5.9$, $-5.99$, $-5.999$, and so on. Then estimate $\lim_{x \to -6} G(x)$. What estimate do you arrive at if you evaluate $G$ at $x = -6.1$, $-6.01$, $-6.001$, ... instead?

b. Support your conclusions in part (a) by graphing $G$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -6$.

c. Find $\lim_{x \to -6} G(x)$ algebraically.

74. Let $h(x) = (x^2 + 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of $h$ at $x = 2.9$, $2.99$, $2.999$, and so on. Then estimate $\lim_{x \to 3} h(x)$. What estimate do you arrive at if you evaluate $h$ at $x = 3.1$, $3.01$, $3.001$, ... instead?

b. Support your conclusions in part (a) by graphing $h$ near $x_0 = 3$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to 3$.

c. Find $\lim_{x \to 3} h(x)$ algebraically.

75. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of $f$ at values of $x$ that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x \to -1} f(x)$.

b. Support your conclusion in part (a) by graphing $f$ near $x_0 = -1$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -1$.

c. Find $\lim_{x \to -1} f(x)$ algebraically.

76. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of $F$ at values of $x$ that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x \to -2} F(x)$.

b. Support your conclusion in part (a) by graphing $F$ near $x_0 = -2$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -2$.

c. Find $\lim_{x \to -2} F(x)$ algebraically.

77. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of $g$ at values of $\theta$ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing $g$ near $\theta_0 = 0$.

78. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of $G$ at values of $t$ that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing $G$ near $t_0 = 0$.

79. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of $f$ at values of $x$ that approach $x_0 = 1$ from above and below. Does $f$ appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing $f$ near $x_0 = 1$.

80. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of $f$ at values of $x$ that approach $x_0 = 0$ from above and below. Does $f$ appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing $f$ near $x_0 = 0$.

81. If $x^4 \leq f(x) \leq x^2$ for $x$ in $[-1, 1]$ and $x^3 \leq f(x) \leq x^4$ for $x < -1$ and $x > 1$, at what points $c$ do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

82. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq 2$ and suppose that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = -5.$$  

Can we conclude anything about the values of $f$, $g$, and $h$ at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \to 2} f(x) = 0$? Give reasons for your answers.
83. If \( \lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1 \), find \( \lim_{x \to 4} f(x) \).

84. If \( \lim_{x \to 2} \frac{f(x)}{x^2} = 1 \), find
   
   a. \( \lim_{x \to 2} f(x) \)
   b. \( \lim_{x \to 2} \frac{f(x)}{x} \)

85. a. If \( \lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 3 \), find \( \lim_{x \to 2} f(x) \).
   
   b. If \( \lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 4 \), find \( \lim_{x \to 2} f(x) \).

86. If \( \lim_{x \to 0} \frac{f(x)}{x^2} = 1 \), find
   
   a. \( \lim_{x \to 0} f(x) \)
   b. \( \lim_{x \to 0} \frac{f(x)}{x} \)

87. a. Graph \( g(x) = x \sin(1/x) \) to estimate \( \lim_{x \to 0} g(x) \), zooming in on the origin as necessary.
   
   b. Confirm your estimate in part (a) with a proof.

88. a. Graph \( h(x) = x^2 \cos(1/x^3) \) to estimate \( \lim_{x \to 0} h(x) \), zooming in on the origin as necessary.
   
   b. Confirm your estimate in part (a) with a proof.

### COMPUTER EXPLORATIONS

In Exercises 89–94, use a CAS to perform the following steps:

a. Plot the function near the point \( x_0 \) being approached.

b. From your plot guess the value of the limit.

89. \( \lim_{x \to 2} \frac{x^4 - 16}{x - 2} \)

90. \( \lim_{x \to 1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2} \)

91. \( \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} \)

92. \( \lim_{x \to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4} \)

93. \( \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \)

94. \( \lim_{x \to 0} \frac{2x^2}{3 - 3 \cos x} \)

## 2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can prove properties given in the preceding section and establish many important limits.

To show that the limit of \( f(x) \) as \( x \to x_0 \) equals the number \( L \), we need to show that the gap between \( f(x) \) and \( L \) can be made “as small as we choose” if \( x \) is kept “close enough” to \( x_0 \). Let us see what this would require if we specified the size of the gap between \( f(x) \) and \( L \).

**EXAMPLE 1** Consider the function \( y = 2x - 1 \) near \( x_0 = 4 \). Intuitively it is clear that \( y \) is close to 7 when \( x \) is close to 4, so \( \lim_{x \to 4} (2x - 1) = 7 \). However, how close to \( x_0 = 4 \) does \( x \) have to be so that \( y = 2x - 1 \) differs from 7 by, say, less than 2 units?

**Solution** We are asked: For what values of \( x \) is \( |y - 7| < 2 \)? To find the answer we first express \( |y - 7| \) in terms of \( x \):

\[
|y - 7| = |(2x - 1) - 7| = |2x - 8|.
\]

The question then becomes: what values of \( x \) satisfy the inequality \( |2x - 8| < 2 \)? To find out, we solve the inequality:

\[
|2x - 8| < 2 \\
-2 < 2x - 8 < 2 \\
6 < 2x < 10 \\
3 < x < 5 \\
-1 < x - 4 < 1.
\]

Keeping \( x \) within 1 unit of \( x_0 = 4 \) will keep \( y \) within 2 units of \( y_0 = 7 \) (Figure 2.15).