Math 106 Calculus 1

Topics for first exam

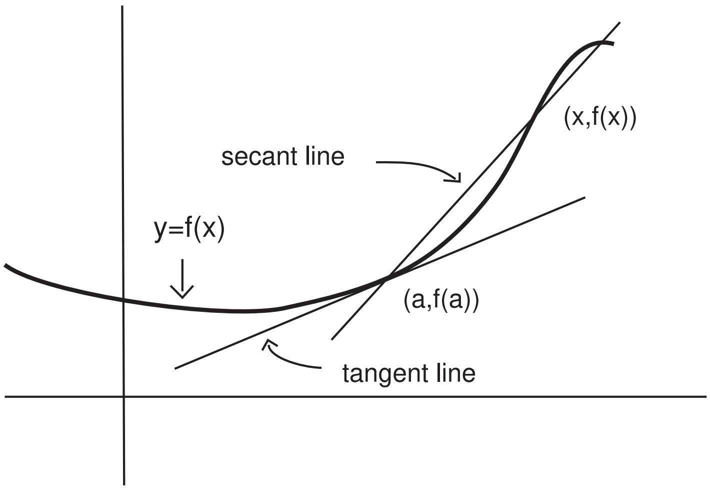
Chapter 2: Limits and Continuity

Rates of change and limits:

Limit of a function f at a point a = the value the function 'should' take at the point
= the value that the points 'near' a tell you f should have at a

$\lim\limits_{x\to a} f(x) = L$ means $f(x)$ is close to L when x is close to (but not <u>equal</u> to) a

Idea: slopes of tangent lines



The closer x is to a, the better the slope of the secant line will approximate the slope of the tangent line.

The slope of the tangent line = limit of slopes of the secant lines (through (a,f(a)))

$\lim\limits_{x\to a} f(x) = L$ does <u>not</u> care what $f(a)$ <u>is</u>; it ignores it

$\lim\limits_{x\to a} f(x)$ need not exist! (function can't make up it's mind?)

Rules for finding limits:

If two functions $f(x)$ and $g(x)$ agree (are equal) for every x near a
(but maybe not <u>at</u> a), then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$

Ex.: $\displaystyle\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2}\frac{(x-1)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{x-1}{x+2} = 1/over4$

If $f(x)\to L$ and $g(x)\to M$ as $x\to a$ (and c is a constant), then
$f(x)+g(x)\to L+M$; $f(x)-g(x)\to L-M$; $cf(x)\to cL$;
$f(x)g(x)\to LM$; and $f(x)/g(x)\to L/M$ <u>provided</u> $M\neq 0$

If $f(x)$ is a polynomial, then $\lim\limits_{x\to x_0} f(x) = f(x_0)$

Basic principle: to solve $\lim\limits_{x\to x_0}$, plug in $x = x_0$!

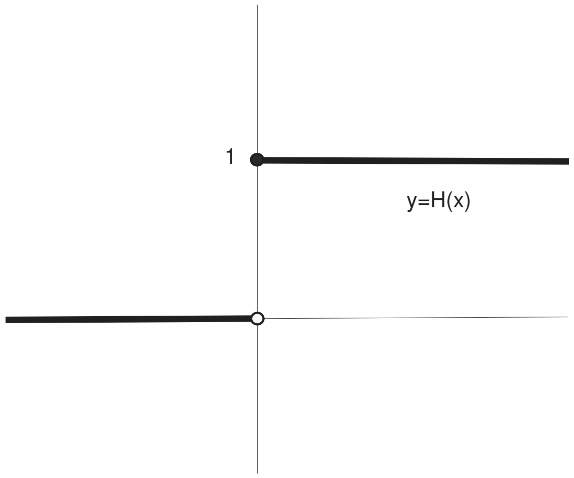
If (and when) you get $0/0$, try something else! (Factor $(x-a)$ out of top and bottom...)
If a function has something like $\sqrt{x}-\sqrt{a}$ in it, try multiplying (top and bottom) with $\sqrt{x}+\sqrt{a}$

(idea: $u=\sqrt{x}, v=\sqrt{a}$, then $x-a = u^2-v^2 = (u-v)(u+v)$.)

Sandwich Theorem: If $f(x) \leq g(x) \leq h(x)$, for all x near a (but not at a), and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

One-sided limits:

Motivation: the Heaviside function



The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, H(0) `wants' to be 0, while on the right, H(0) `wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the `one-sided' limits DO exist.

Limit from the right: $\lim_{x \to a^+} f(x) = L$ means $f(x)$ is close to L
when x is close to, and bigger than, a

Limit from the left: $\lim_{x \to a^-} f(x) = M$ means $f(x)$ is close to M
when x is close to, and smaller than, a

$\lim_{x \to a} f(x) = L$ then means $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$
(i.e., both one-sided limits exist, and are equal)

Limits at infinity / infinite limits:

∞ represents something bigger than any number we can think of.

$\lim_{x \to \infty} f(x) = L$ means $f(x)$ is close of L when x is really large.

$\lim_{x \to -\infty} f(x) = M$ means $f(x)$ is close of M when x is really large and *negative*.

Basic fact: $\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$

More complicated functions: divide by the highest power of x in the denominator.
$f(x), g(x)$ polynomials, degree of $f = n$, degree of $g = m$

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0 \text{ if } n < m$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = (\text{coeff of highest power in } f)/(\text{coeff of highest power in } g) \text{ if } n = m$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \pm\infty \text{ if } n > m$$

$\lim_{x \to a} f(x) = \infty$ means $f(x)$ gets really large as x gets close to a

Also have $\lim_{x\to a} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = \infty$; $\lim_{x\to a^-} f(x) = \infty$; etc....

Typically, an infinite limit occurs where the denomenator of $f(x)$ is zero
(although not always)

Asymptotes:

The line $y = a$ is a horizontal asymptote for a function f if
$\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is equal to a.
I.e., the graph of f gets really close to $y = a$ as $x \to \infty$ or $a \to -\infty$

The line $x = b$ is a vertical asymptote for f if $f \to \pm\infty$ as $x \to b$ from the right or left.
If numerator and denomenator of a rational function have no common roots, then vertical asymptotes = roots of denom.

Continuity:

A function f is <u>continuous</u> (cts) <u>at</u> <u>a</u> if $\lim_{x\to a} f(x) = f(a)$
This means: (1) $\lim_{x\to a} f(x)$ exists ; (2) $f(a)$ exists ; and
(3) they're equal.
Limit theorems say (sum, difference, product, quotient) of cts functions are cts.
Polynomials are continuous at every point;
rational functions are continuous except where denom=0.
Points where a function is not continuous are called <u>discontinuities</u>
Four flavors:
removable: both one-sided limits are the same
jump: one-sided limts exist, not the same
infinite: one or both one-sided limits is ∞ or $-\infty$
oscillating: one or both one-sided limits DNE

Intermediate Value Theorem:
If $f(x)$ is cts at every point in an interval $[a, b]$, and M is between $f(a)$ and $f(b)$, then there is (at least one) c between a and b so that $f(c) = M$.

Application: finding roots of polynomials

Tangent lines:

Slope of tangent line = limit of slopes of secant lines; at $(a, f(a)$:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Notation: call this limit $f'(a)$ = derivative of f at a

Different formulation: $h = x - a$, $x = a + h$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \text{limit of } \textit{difference quotient}$$

If $y = f(x)$ = position at 'time' x, then difference quotient = average velocity;
limit = instantaneous velocity.

Chapter 3: Derivatives

The derivative of a function:

derivative = limit of difference quotient (two flavors: $h \to 0$, $x \to a$)

If $f'(a)$ exists, we say f is <u>differentiable</u> at a

Fact: f differentiable (diff'ble) at a, then f cts at a

Using $h \to 0$ notation: replace a with x (= variable), get $f'(x)$ = <u>new</u> <u>function</u>

Or: $f'(x) = \lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x}$

$f'(x)$ = the derivative of f = function whose values are the slopes of the tangent lines to the graph of $y{=}f(x)$. Domain = every point where the limit exists

Notation:

$f'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}(f(x)) = \dfrac{df}{dx} = y' = D_x f = Df = (f(x))'$

Differentiation rules:

$\dfrac{d}{dx}(\text{constant}) = 0 \qquad \dfrac{d}{dx}(x) = 1$

$(f(x){+}g(x))' = (f(x))'{+}\ (g(x))' \qquad (f(x){-}g(x))'{=}\ (f(x))'{-}\ (g(x))'$

$(\mathrm{c}f(x))'{=}\ \mathrm{c}(f(x))'$

$(f(x)g(x))'{=}\ (f(x))'g(x){+}\ f(x)(g(x))' \qquad (\dfrac{f(x)}{g(x)})'{=}\ \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$(x^n)'{=}\ nx^{n-1}$, for n = ~~natural number~~ ~~integer~~ rational number

$(a^x)'{=}\ K \cdot a^x$, where $K = \dfrac{d}{dx}(a^x)\Big|_{x=0}$

[[$(1/g(x))'{=}\ {-}g'(x)/(g(x))^2$]]

Higher derivatives:

$f'(x)$ is 'just' a <u>function</u>, so we can take its derivative!

$(f'(x))'{=}\ f''(x) \quad (= y'' = \dfrac{d^2y}{dx^2} = \dfrac{d^2f}{dx^2})$

= second derivative of f

Keep going! $f'''(x)$ = 3rd derivative, $f^{(n)}(x)$ = nth derivative

Rates of change

Physical interpretation:

$f(t){=}$ position at time t

$f'(t){=}$ rate of change of position = velocity

$f''(t){=}$ rate of change of velocity = acceleration

$|f'(t)|$ = speed

Basic principle: for object to change direction (velocity changes sign),

$f'(t){=}\ 0$ somewhere (IVT!)

Examples:

Free-fall: object falling near earth; $s(t) = s_0 + v_0 t - \dfrac{g}{2}t^2$

$s_0 = s(0)$ = initial position; v_0 = initial velocity; $g{=}$ acceleration due to gravity

Economics:
$C(x)$ = cost of making x objects; $R(x)$ = revenue from selling x objects;
$P = R - C$ = profit
$C'(x)$ = marginal cost = cost of making 'one more' object
$R'(x)$ = marginal revenue ; profit is maximized when $P'(x) = 0$;
i.e., $R'(x) = C'(x)$

Derivatives of trigonometric functions

Basic limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$; everything else comes from this! $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

Note: this uses radian measure!

$$\lim_{x\to 0} \frac{\sin(bx)}{x} = \lim_{x\to 0} b\frac{\sin(bx)}{bx} = \lim_{u\to 0} b\frac{\sin(u)}{u} = b$$

Then we get:
$(\sin x)' = \cos x$ $(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$ $(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$ $(\csc x)' = -\csc x \cot x$