Commutators, Derivations, and the Heisenberg Commutation Relation

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**Ingredients**: Hilbert space $H$, self-adjoint operator $D$, and initial state $\psi(0) \in H$.

**Question**: What do we expect to measure for an observable $x \in B(H)$ at time $t$?

**Expected Value**:

$$\langle x\psi(t), \psi(t) \rangle = \langle xe^{-itD}\psi(0), e^{-itD}\psi(0) \rangle = \langle e^{itD}xe^{-itD}\psi(0), \psi(0) \rangle.$$  

**Baker-Campbell-Hausdorff Formula**: For matrices $D$ and $x$,

$$e^{itD}xe^{-itD} = x + t[iD, x] + \frac{t^2}{2!}[iD, [iD, x]] + \frac{t^3}{3!}[iD, [iD, [iD, x]]] + ...$$
Baker-Campbell-Hausdorff Formula: For operators $D$ and $x$ on a Hilbert space,

$$e^{itD}xe^{-itD} = x + t[iD, x] + \frac{t^2}{2!} [iD, [iD, x]] + \frac{t^3}{3!} [iD, [iD, [iD, x]]] + \ldots$$

If $A : \text{dom } A \to H$ and $B : \text{dom } B \to H$ are linear operators, then

$$\text{dom } [A, B] = \{ h \in \text{dom } A \cap \text{dom } B : Ah \in \text{dom } B \text{ and } Bh \in \text{dom } A \}.$$  

- It’s not necessarily the case that $\text{dom } A \cap \text{dom } B$ is dense in $H$.
- Even if $\text{dom } [A, B]$ is dense in $H$, the domain of iterated commutators might not be.

Goal: Recognize $[iD, \ldots [iD, x]]$ as $\delta^n(x)$ on a suitable dense subspace of $H$. 

$n$ times
1. Introduce weakly-defined derivation $\delta_D$ and some of its properties

2. Formally relate $\delta^n_D(x)$ to the iterated commutator $[iD, \ldots, [iD, x]]$ \(n\) times

3. Show $\delta_D$ has kernel stabilization: $\ker \delta^n_D = \ker \delta_D$ for all $n \in \mathbb{N}$

4. Applications
   - Kernel stabilization of a family of abstract derivations
   - Heisenberg Commutation Relation
Weak $D$-differentiability and the derivation $\delta_D$

For each $t \in \mathbb{R}$ and $x \in B(H)$, define $\alpha_t(x) := e^{itD}xe^{-itD}$.

**Definition**

An operator $x \in B(H)$ is **weakly $D$-differentiable** if for every $h, k \in H$, the function

$$t \mapsto \langle \alpha_t(x)h, k \rangle$$

is continuously differentiable. Equivalently, there exists $y \in B(H)$ such that for every $h, k \in H$,

$$\lim_{t \to 0} \left| \left| \langle \left( \frac{\alpha_t(x) - x}{t} \right) h, k \rangle - \langle yh, k \rangle \right| \right| = 0.$$

Define $\delta_D(x) := y$. 
Properties of $\delta_D$

Let $\text{dom } \delta_D$ denote the set of all weakly $D$-differentiable operators.

**Theorem (Christensen, ’15)**

- The linear operator $\delta_D$ is a $\ast$-derivation on $\text{dom } \delta_D$.
- The graph of $\delta_D$ is strong operator topology (SOT)-closed.
- The domain of $\delta_D$ is SOT-dense in $B(H)$.

For each $n \in \mathbb{N}$, $\text{dom } \delta_D^n = \{ x \in \text{dom } \delta_D^{n-1} : \delta_D^{n-1}(x) \in \text{dom } \delta_D \}$.

**Theorem (I., ’18)**

For all $n \in \mathbb{N}$, $\text{dom } \delta_D^n$ is SOT-dense in $B(H)$.

**Theorem (I., ’18)**

The set of analytic vectors for $\delta_D$ are SOT-dense in $B(H)$.
Iterated commutators and powers of $\delta_D$

**Definition**

An operator $x \in B(H)$ is $n$-times weakly $D$-differentiable (denoted $x \in \text{dom } \delta^n_D$) if and only if for every $h, k \in H$, the map $t \mapsto \langle \alpha_t(x)h, k \rangle$ is $n$-times continuously differentiable.

**Theorem (Christensen ’15)**

An operator $x \in B(H)$ is $n$-times weakly $D$-differentiable if and only if there exists a core $C \subset \text{dom } D$ for $D$ such that

$$[iD, \ldots, [iD, x]]$$

$k$ times

is well-defined and bounded on $C$ for each $k \leq n$. When this holds,

$$\delta^n_D(x)|_C = [iD, \ldots, [iD, x]].$$

$n$ times
Motivating example for kernel stabilization

Example

Let $H = \ell^2(\mathbb{Z})$. Define $(Df)(j) =jf(j)$ for $f \in \text{dom } D$, where

$$\text{dom } D := \left\{ g \in \ell^2(\mathbb{Z}) : \sum_{n \in \mathbb{Z}} |ng(n)|^2 < \infty \right\}.$$

Question: Given $n \in \mathbb{N}$, what is $\ker \delta^n_D$? For $x \in \text{dom } \delta^n_D$,

$$\delta_D(x)_{rc} = [iD, x]_{rc} = i(Dx - xD)_{rc} = i(rx_{rc} - x_{rc} c) = i(r - c)x_{rc},$$

so

$$\delta^n_D(x)_{rc} = i^n(r - c)^n x_{rc}.$$
Motivating example for kernel stabilization

Example (What is ker $\delta^n_D$?)

$x \in \ker \delta^n_D \iff \delta^n_D(x)_{rc} = 0 \text{ for all } r, c \in \mathbb{Z}$

$\iff i^n (r - c)^n x_{rc} = 0 \text{ for all } r, c \in \mathbb{Z}$

$\iff x_{rc} = 0 \text{ for all } r, c \in \mathbb{Z} \text{ where } r \neq c$

$\iff x \text{ is diagonal in the standard orthonormal basis.}$

Thus, ker $\delta^n_D$ is precisely the von Neumann algebra of diagonal multiplication operators on $\ell^2(\mathbb{Z})$, 

$$\ell^\infty(\mathbb{Z}) \hookrightarrow B(\ell^2(\mathbb{Z})).$$
General kernel stabilization of $\delta_D$

**Question:** In the example, what is the relationship between $D$ and the diagonal multiplication operators $\ell^\infty(\mathbb{Z})$? Let $\mathcal{M}_D$ denote the von Neumann algebra generated by the set of spectral projections for $D$.

**Example**

Note that $\mathcal{M}_D = \ell^\infty(\mathbb{Z})$, so for all $n \in \mathbb{N}$,

$$\ker \delta^n_D = \ell^\infty(\mathbb{Z}) = \mathcal{M}_D.$$

Is this true in general? No: $\mathcal{M}_D \subseteq \ker \delta_D \subseteq \mathcal{M}'_D$. Note that $\ell^\infty(\mathbb{Z}) = \ell^\infty(\mathbb{Z})'$, so in fact

$$\ker \delta^n_D = \mathcal{M}'_D.$$

**Theorem (I., 2017)**

For any self-adjoint operator $D$, $\ker \delta^n_D = \mathcal{M}'_D$ for all $n \in \mathbb{N}$. 

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Application 1: Unbounded derivations on $C^*$-algebras

**Theorem (Bratteli-Robinson, ’74)**

Let $\mathcal{A}$ be a $C^*$-algebra, and let $\delta$ be a (possibly unbounded) $*$-derivation of $\mathcal{A}$ with the properties:

- there exists a state $\omega$ on $\mathcal{A}$ which generates a faithful cyclic representation $(\pi_\omega, H_\omega, f_\omega)$ of $\mathcal{A}$ such that
  \[ \omega(\delta(a)) = 0 \]
  for all $a \in \text{dom} \, \delta$, and
- $\delta$ has a norm-dense set of analytic elements in $\mathcal{A}$.

Then there exists an essentially self-adjoint operator $S$ on $H_\omega$ such that

\[ \text{dom} \, S = \{ h \in H_\omega : h = \pi(a)f_\omega \text{ for some } a \in \mathcal{A} \} \]

and for all $h \in \text{dom} \, S$,

\[ \pi(\delta(a))h = [iS, \pi(a)]h. \]
Corollary (I., 2017)

If $\pi, \mathcal{A},$ and $\delta$ are as in Bratteli & Robinson’s theorem, then $\ker \delta^n = \ker \delta$.

Proof: Let $D := \overline{S}$. Show that $\delta_D \circ \pi$ extends $\pi \circ \delta$ on the analytic vectors $A_\delta$ for $\delta$. The derivation $\delta$ on $\mathcal{A}$ is inherited from its represented extension $\delta_D$.

Future Directions:

1. For an arbitrary derivation, does kernel stabilization guarantee an invariant state?

2. What about some sort of density of analytic vectors?
Application 2: The Heisenberg Commutation Relation

Definition

Two self-adjoint operators $A : \text{dom } A \rightarrow H$ and $B : \text{dom } B \rightarrow H$ satisfy the HCR if

$$[A, B]k = ik$$

for all $k$ in some dense subspace $K \subseteq H$ which is contained in $\text{dom } [A, B]$.

Facts.

1. No two bounded self-adjoint operators can satisfy the HCR.
2. There are examples of two unbounded operators satisfying the HCR.
3. There are examples of a bounded and unbounded operator pair satisfying the HCR.
Example (The Schrödinger Pair)

Let $H = L^2(\mathbb{R})$, $P = -i \frac{d}{dx}$ with

$$\text{dom } P = \{ f \in L^2(\mathbb{R}) : f \text{ abs. cts. and } f' \in L^2(\mathbb{R}) \}$$

and $Q = M_x$ with

$$\text{dom } Q = \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |xf(x)|^2 \, dx < \infty \right\}.$$

Then $[Q, P]k = ik$ for all $k \in S(\mathbb{R})$;

$$S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \| Q^m P^n f \|_\infty < \infty \ \forall m, n \in \mathbb{N} \}.$$

Note: The Schwartz space $S(\mathbb{R})$ is contained in $\text{dom } [Q, P]$ and is dense in $L^2(\mathbb{R})$. In fact, $S(\mathbb{R})$ is a core for both $P$ and $Q$. Note both $P$ and $Q$ are unbounded.
**Example**

Let $H = L^2[0, 1]$, $p = -i \frac{d}{dx}$ with

$$\text{dom } p = \{ f \in L^2[0, 1] : f \text{ abs. cts., } f' \in L^2[0, 1], \ f(0) = f(1) \},$$

and let $q = M_x \in B(L^2[0, 1])$. Then $[q, p]k = ik$ for all

$$k \in K := \{ f \in \text{dom } p : f(0) = f(1) = 0 \} \subseteq \text{dom } [q, p].$$

**Note:** $K$ is contained in $\text{dom } [p, q]$ and is dense in $H$. However, $K$ is not a core for $p$.

**Question:** When are two self-adjoint operators which satisfy the Heisenberg Commutation Relation necessarily both unbounded?
Theorem (I., 2018)

Let $A$ and $B$ be self-adjoint operators which satisfy the HCR on a dense subspace $K \subseteq H$. If $K$ is a core for $A$ and $B$, then both $A$ and $B$ must be unbounded.

Proof: Suppose $A \in B(H)$, $K \subseteq \text{dom} \ [A, B]$ is a core for $B$ ($A$ is bounded!), and $[A, B]k = ik$ for all $k \in K$. Note:

$$[A, B]k = ik \iff [iB, A]k = k.$$

By Christensen, $A \in \text{dom} \ \delta_B^2$ and

$$\delta_B^2(A)|_K = [iB, [iB, A]] = [iB, I] = 0.$$

Therefore, $A \in \ker \delta_B^2 = \ker \delta_B$ by kernel stabilization. But then $\delta_B(A)|_K = [iB, A] = 0$, a contradiction to HCR. Hence, $A \notin B(H)$. $\square$
Thank you!