Homological criteria for regular homomorphisms and for locally complete intersection homomorphisms

Luchezar L. Avramov* and Srikanth Iyengar†

January 11, 2001

Introduction

Regularity and complete intersection are properties of commutative noetherian rings, motivated by geometry, but often best approached through homological methods. For an illustration, it suffices to recall that no non-homological proof is known for the stability of these properties under such a basic operation as localization.

It has long been known that 'nice' algebras have 'simple' homology. For flat algebras of finite type a famous example is provided by Hochschild, Kostant, and Rosenberg [28]: The Hochschild homology of a smooth algebra is the exterior algebra on a finite projective module, namely, the module of Kähler differentials.

Two converses have been obtained over the last ten years, as the result of work of several authors—one in terms of vanishing of certain Hochschild homology modules, the other through the finite generation of the Hochschild homology algebra. In Section 5 we deduce these results from theorems characterizing vanishing and finite generation within the more general framework of Cartan-Eilenberg homology of supplemented algebras.

The vanishing theorem for Cartan-Eilenberg homology is due to Rodicio [39]. A complete proof is given in Section 3, with short new arguments for the preliminary results. In Section 4 we discuss the key ideas going into the theorem on finite generation. The proof given in our paper [12] used a recent result on the vanishing of André-Quillen homology of algebras of finite flat dimension, cf. [9]: this suffices for the application to Hochschild homology. In [13] we lifted the restriction, after proving a special case of a long open conjecture of Quillen. This material is discussed in Section 2. It is preceded by a review, in Section 1, of earlier results on the vanishing of André-Quillen homology and their applications to regular homomorphisms and l.c.i. homomorphisms.

In the final count, the results on the structure of homomorphisms of commutative rings reported below are based on an interplay of several homology theories, abelian and simplicial.

*Partly supported by a grant from the N.S.F.

†Supported by a grant from the E.P.S.R.C.; this author thanks Purdue University for its hospitality during a visit when this paper was written.

1 Smooth, regular, and l.c.i. homomorphisms

In this paper all rings are commutative.

Let S be a ring. A sequence $\boldsymbol{y} = y_1, \ldots, y_c$ of elements of S is called *regular* if $(\boldsymbol{y}) \neq S$ and y_i is a non-zero-divisor on $S/(y_1, \ldots, y_{i-1})$ for $i = 1, \ldots, c$. An ideal I in S is *locally generated by a regular sequence* if for each prime ideal $\mathfrak{q}$ of S with $I \subseteq \mathfrak{q}$ the ideal $I_\mathfrak{q}$ is generated by an $S_\mathfrak{q}$-regular sequence.

We say that $(S, \mathfrak{n}, k)$ is a local ring if S is a noetherian ring with unique maximal ideal $\mathfrak{n}$ and residue field $k = S/\mathfrak{n}$. If an ideal I in a local ring is generated by a regular sequence, then every minimal set of generators of I constitutes a regular sequence.

A local ring $(S, \mathfrak{n}, k)$ is *regular* if $\mathfrak{n}$ has a set of generators that form an S-regular sequence. A ring S is said to be regular if it is noetherian and the local ring $S_\mathfrak{n}$ is regular for each maximal ideal $\mathfrak{n}$ of S. This property is inherited by all localizations of S, cf. e.g. [20] or [34].

We discuss relative versions of the notions of regularity. The earliest one is that of smoothness. We introduce it through the Jacobian criterion, for it best reflects the geometric origin of the concept. To this end, recall that the module of Kähler differentials of a K-algebra S is defined by the equality $\Omega_{S|K} = I/I^2$, where I is the kernel of the ring homomorphism

$$\mu_K^S \colon S \otimes_K S \to S \quad \text{where} \quad s' \otimes s'' \mapsto s's'' .$$

Let S be *essentially of finite type* over K, that is, a localization of a K-algebra of finite type. If $\mathfrak{q}$ is a prime ideal of S, and $k(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ is the residue field of $S_\mathfrak{q}$, then S is *smooth* over K at $\mathfrak{q}$ if the K-module $S_\mathfrak{q}$ is flat and the $S_\mathfrak{q}$-module $(\Omega_{S|K})_\mathfrak{q}$ is free of rank $\dim\left(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}\right) + \operatorname{tr}\deg_{k(\mathfrak{p})}(k(\mathfrak{q}))$, where $\mathfrak{p} = \mathfrak{q} \cap K$. The K-algebra S is *smooth* if it is smooth at each $\mathfrak{q} \in \operatorname{Spec} S$. When K and S are fields, smoothness is equivalent to separability.

Theorem 1.1 *If K is a noetherian ring and S a flat K-algebra essentially of finite type, then the following conditions are equivalent.*

(i) *S is smooth over K.*

(ii) *$S \otimes_K L$ is regular for each K-algebra L that is a field and is essentially of finite type.*

(iii) *$\operatorname{Ker}(\mu_K^S)$ is locally generated by a regular sequence.*

If, in addition, the characteristic[1] *of S is 0, then they are also equivalent to*

(i′) *$\Omega_{S|K}$ is projective over S.*

For the equivalence of (i) (respectively, (i′)) and (ii), confer [21, §17.5] or [29, §8]. When K is a perfect field, (ii) $\iff$ (iii) is proved in [28]. It seems that all published proofs of the general case use at some point André-Quillen homology. At the end of the section we sketch the argument from [4], after reviewing basic properties of that theory.

Conditions (i) and (iii) in the theorem above critically depend on the hypothesis that the algebra S is essentially of finite type. On the other hand, condition (ii) does not require such a finiteness assumptions, and may be used to define a notion of regularity for arbitrary maps. This is Grothendieck's approach [21, §6]: He defines a homomorphism $\varphi\colon R \to S$ of

[1] A ring has characteristic 0 if it contains the field of rational numbers as a subring.

noetherian rings to be *regular* (respectively, *normal*, *Cohen-Macaulay*, etc.), if S is flat over R and the ring $S \otimes_R L$ is regular (respectively, normal, Cohen-Macaulay, etc.) whenever $R \to L$ is a homomorphism essentially of finite type and L is a field.

The definitive criterion for regularity in terms of André-Quillen homology appears in [2, (S.30)]; it sums up results of André [1], Grothendieck [21], Harrison [25], and Quillen [36].

Theorem 1.2 *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.*
The following conditions are equivalent.

(i) *φ is regular.*

(ii) $\mathrm{D}_1(S|R;-) = 0$.

(iii) $\mathrm{D}_n(S|R;-) = 0$ *for all* $n \geq 1$.

Now we move on to the complete intersection property.

By Cohen's Structure Theorem, for every local ring $(S, \mathfrak{n}, k)$ the $\mathfrak{n}$-adic completion $\widehat{S}$ has a *Cohen presentation* $\widehat{S} \cong Q/J$, with Q a complete regular local ring; this is one of the reasons regular local rings play a crucial role in commutative algebra. A local ring S is *complete intersection* if in some Cohen presentation of $\widehat{S}$ the defining ideal J can be generated by a Q-regular sequence; when this is the case, every Cohen presentation of $\widehat{S}$ has this property. A ring S is a *complete intersection* if $S_{\mathfrak{n}}$ is a complete intersection for each maximal ideal $\mathfrak{n} \in \operatorname{Spec} S$. If S is a complete intersection, then so is $S_{\mathfrak{q}}$ for each $\mathfrak{q} \in \operatorname{Spec} S$ cf. [6].

A notion of complete intersection homomorphism can be defined by Grothendieck's approach, but it is not broad enough to accommodate the most used version of that concept: homomorphisms $\varphi\colon R \to S$ admitting a factorization $R \xrightarrow{\iota} R[x] \xrightarrow{\pi} S$, where $R[x]$ is a polynomial ring and π is a surjection whose kernel is locally generated by a regular sequence. The last notion has its own limitation—it can only be applied to maps of finite type.

A concept that makes no *a priori* hypotheses on the homomorphism is introduced in [9], using the following construction of Avramov, Foxby, and Herzog [10, (1.1)]: For each local homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ there exists a commutative diagram

$$\begin{array}{ccc} & R' & \\ {\scriptstyle\dot{\varphi}}\nearrow & & \searrow{\scriptstyle\varphi'} \\ R & \xrightarrow{\grave{\varphi}} & \widehat{S} \end{array}$$

of local homomorphisms where $\grave{\varphi}\colon R \to \widehat{S}$ is the composition of φ with the canonical inclusion $S \to \widehat{S}$, the R-module R' is flat, the ring R' is complete, the ring $R'/\mathfrak{m}R'$ is regular, and the map φ' is surjective. Any such diagram is called a *Cohen factorization* of $\grave{\varphi}$.

A homomorphism of rings $\varphi\colon R \to S$ is *complete intersection* at a prime ideal $\mathfrak{q}$ of S if the induced local homomorphism $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ has a Cohen factorization in which the kernel of the surjective map is generated by a regular sequence; this property does not depend on the choice of Cohen factorization, cf. [10] or [9]. The homomorphism φ is *locally complete intersection*, abbreviated to *l.c.i.*, if it is complete intersection at each $\mathfrak{q} \in \operatorname{Spec} S$.

This definition covers the special cases encountered earlier cf. [9]: A ring S is complete intersection if and only if the structure map $\mathbb{Z} \to S$ is l.c.i.; a flat map is l.c.i. if and only if it is so in the sense of Grothendieck; when S is of finite type with a factorization $\varphi = \pi\iota$ as above, then φ is l.c.i. precisely when $\operatorname{Ker}(\pi)$ is locally generated by a regular sequence.

Vanishing of André-Quillen homology is linked to the l.c.i. property by the next result. For homomorphisms essentially of finite type it is proved by Lichtenbaum and Schlessinger [32], André [1], and Quillen [36]; the general case is established in [9].

Theorem 1.3 *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.*
The following conditions are equivalent.

(i) *$\varphi\colon R \to S$ is locally complete intersection.*

(ii) $\mathrm{D}_2(S\,|\,R; -) = 0$.

(iii) $\mathrm{D}_n(S\,|\,R; -) = 0$ *for* $n \geq 2$.

(iv) $\mathrm{D}_3(S\,|\,R; -) = 0$ *and the R-module $S_{\mathfrak{q}}$ has a finite flat dimension for each* $\mathfrak{q} \in \operatorname{Spec} S$.

The characterizations of regularity and complete intersection in terms of André-Quillen homology have important applications to the study of these classes of homomorphisms. Some of their properties, such as flat base change, can be obtained either as formal consequences of the behavior of the cohomology theory, cf. 1.5, or through more traditional techniques. Other properties are accessible at present only from the characterizations given above.

A case in point is localization. The problem is to find conditions on a local ring $(R, \mathfrak{m}, k)$ that would guarantee that if a local homomorphism $\varphi\colon R \to (S, \mathfrak{n}, l)$ has a certain property at $\mathfrak{n}$, then it has the same property at each $\mathfrak{q} \in \operatorname{Spec} S$. When φ is flat and essentially of finite type, it is not difficult to see that if the induced homomorphism $k \otimes_R \varphi\colon k \to S/\mathfrak{m}S$ is regular or complete intersection, then the homomorphism φ is regular or l.c.i.

However, the analogous statement for the canonical homomorphism $R \to \widehat{R}$ fails if the formal fibers of R lack the corresponding property, and examples are known of rings for which this occurs. Localization theorems from [3] and [9] prove that this is the only obstruction.

Theorem 1.4 *Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism.*

If φ is flat, the homomorphism $k \to S/\mathfrak{m}S$ is regular, and so is the homomorphism $k(\mathfrak{p}) \to k(\mathfrak{p}) \otimes_R \widehat{R}$ for every $\mathfrak{p} \in \operatorname{Spec} R$, then φ is regular.

If φ is complete intersection at $\mathfrak{n}$, and the ring $k(\mathfrak{p}) \otimes_R \widehat{R}$ is complete intersection for each $\mathfrak{p} \in \operatorname{Spec} R$, then φ is l.c.i.

Next we list some basic properties of André-Quillen homology, cf. [2], [36].

Remark 1.5 Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. For each S-module N and every $n \in \mathbb{Z}$ an S-module $\mathrm{D}_n(S\,|\,R; N)$ is defined with the following properties.

Functoriality. The sequence $\{\mathrm{D}_n(S\,|\,R; -)\}_{n \in \mathbb{Z}}$ is a homological functor on the category of S-modules, and there are isomorphisms of functors

$$\mathrm{D}_0(S\,|\,R; -) \cong \Omega_{S\,|\,R} \otimes_S - \qquad \text{and} \qquad \mathrm{D}_n(S\,|\,R; -) = 0 \quad \text{for all} \quad n < 0\,.$$

Vanishing. If X is a set of variables over R, then

$$\mathrm{D}_n(R[X]\,|\,R; -) = 0 \quad \text{for all} \quad n \geq 1\,.$$

Localization. For each $\mathfrak{q} \in \operatorname{Spec} S$, there are isomorphisms of functors

$$\mathrm{D}_n(S\,|\,R;-)_{\mathfrak{q}} \cong \mathrm{D}_n(S_{\mathfrak{q}}\,|\,R;-_{\mathfrak{q}}) \cong \mathrm{D}_n(S_{\mathfrak{q}}\,|\,R_{\mathfrak{q}\cap R};-_{\mathfrak{q}}) \quad \text{for all} \quad n \in \mathbb{Z}\,.$$

Base change. If $\varphi = R \otimes_{R'} \varphi'$, where $R \leftarrow R' \xrightarrow{\varphi'} S'$ are homomorphisms of rings one of which is flat, then there are isomorphisms of functors

$$\mathrm{D}_n(S\,|\,R;-) \cong \mathrm{D}_n(S'\,|\,R';-) \quad \text{for all} \quad n \in \mathbb{Z}\,.$$

Transitivity. A homomorphism of rings $Q \to R$ induces an exact sequence of functors

$$\cdots \to \mathrm{D}_{n+1}(S\,|\,R;-) \to \mathrm{D}_n(R\,|\,Q;-) \to \mathrm{D}_n(S\,|\,Q;-) \to \mathrm{D}_n(S\,|\,R;-) \to \cdots\,.$$

This generalizes the "first fundamental exact sequence" $\Omega_{R|Q} \otimes_R S \to \Omega_{S|Q} \to \Omega_{S|R} \to 0\,.$

Low dimensions. If φ is surjective and $I = \operatorname{Ker}(\varphi)$, then there exist morphisms of functors $\operatorname{Tor}_n^R(S,-) \to \mathrm{D}_n(S\,|\,R;-)$ for all $n \in \mathbb{Z}$, inducing isomorphisms

$$\mathrm{D}_1(S\,|\,R;-) \cong \operatorname{Tor}_1^R(S,-) \cong (I/I^2) \otimes_S - \;;$$

$$\mathrm{D}_2(S\,|\,R;-) \cong \frac{\operatorname{Tor}_2^R(S,-)}{\operatorname{Tor}_1^R(S,S) \cdot \operatorname{Tor}_1^R(S,-)}\,.$$

The morphisms above arise as edge homomorphisms in a spectral sequence converging to the torsion functors, with second page determined by the André-Quillen functors.

The transitivity sequence is a salient feature of André-Quillen homology, distinguishing it from other homology theories for commutative algebras. It reflects the homotopical nature of the André-Quillen theory, whose construction is based on simplicial commutative rings, cf. [1], [35]. Proofs of the properties discussed above depend on specifics of the construction. Once made available, these properties constitute a powerful tool that is sufficiently flexible and user-friendly. Illustrations of such 'naïve' usage are given below.

Remark 1.6 If $Q \to R \to S$ are homomorphisms of rings, and m is an integer such that $\mathrm{D}_n(S\,|\,Q;-) = 0$ for all $n \geq m$, then transitivity yields isomorphisms of functors

$$\mathrm{D}_{n+1}(S\,|\,R;-) \cong \mathrm{D}_n(R\,|\,Q;-) \quad \text{for all} \quad n \geq m\,.$$

Proof (of the equivalence of (ii) and (iii) in Theorem 1.1) By the vanishing property, $\mathrm{D}_n(S\,|\,S;-) = 0$ for all n. Remark 1.6 applied to the maps $\psi\colon S \to S \otimes_K S$, given by $\psi(s) = s \otimes 1$, and $\mu_K^S\colon S \otimes_K S \to S$ yields $\mathrm{D}_2(S\,|\,S \otimes_K S;-) \cong \mathrm{D}_1(S \otimes_K S\,|\,S;-)$. From flat base change we get $\mathrm{D}_1(S \otimes_K S\,|\,S;-) \cong \mathrm{D}_1(S\,|\,K;-)$. By concatenation we obtain

$$\mathrm{D}_2(S\,|\,S \otimes_K S;-) \cong \mathrm{D}_1(S\,|\,K;-)\,.$$

By Theorem 1.2, the K-algebra S is smooth if and only if the functor on the right hand side vanishes. By Theorem 1.3, the functor on the left hand side vanishes if and only if $\operatorname{Ker}\mu_K^S$ is locally generated by a regular sequence. We have the desired equivalence.

□

2 Vanishing of higher André-Quillen homology

Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.

The flat dimension of S over R may be defined by the formula

$$\operatorname{fd}_R S = \sup\{n \in \mathbb{Z} \mid \operatorname{Tor}_n^R(S, -) \neq 0\}\,.$$

Along similar lines, we introduce the *André-Quillen dimension* of S over R by

$$\text{AQ-dim}_R\, S = \sup\{n \in \mathbb{Z} \mid \mathrm{D}_n(S\,|\,R; -) \neq 0\}\,.$$

In this language, Theorems 1.2 and 1.3 describe the algebras of André-Quillen dimension 0 and 1. When R is a local ring and S is its residue field, the invariant $\text{AQ-dim}_R\, S$ is considered by André [1, §28] under the name *simplicial dimension* of R.

We say that the flat dimension, or the André-Quillen dimension, of S over R is *locally finite* if the corresponding dimension of $S_{\mathfrak{q}}$ over R is finite for each $\mathfrak{q} \in \operatorname{Spec} S$.

In this section we discuss finiteness of André-Quillen dimension. The first result, from [9], describes the algebras whose flat dimension and André-Quillen dimension are both finite.

Theorem 2.1 *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.*
The following conditions are equivalent.

(i) *φ is locally complete intersection.*

(ii) *$\text{AQ-dim}_R\, S$ is locally finite and $\operatorname{fd}_R S$ is locally finite.*

If S has characteristic 0, then they are also equivalent to

(iii) *$\mathrm{D}_m(S\,|\,R; -) = 0$ for some integer $m \geq 2$ and $\operatorname{fd}_R S$ is locally finite.*

Theorem 1.3 contains (i) $\implies$ (ii); the converse had been conjectured by Quillen [36, (5.7)], at least when φ is essentially of finite type. Without assumptions on the flat dimension (but retaining the hypothesis of finite type) Quillen [36, (5.6)] made the following

Conjecture 2.2 If the AQ-dimension of S over R is locally finite, then $\text{AQ-dim}_R\, S \leq 2$.

Remark 2.3 Results in [2, Suppl.] and [9, §1] show that to prove the conjecture it suffices to do it for all surjective homomorphisms of complete local rings.

Example 2.4 The conjecture holds if R or S is complete intersection.

For instance, if S has this property, then (as noted above) the structure map $\mathbb{Z} \to S$ is l.c.i., so $\mathrm{D}_n(S|\mathbb{Z}; -) = 0$ for $n \geq 2$ by Theorem 2.1. Remark 1.6 applied to $\mathbb{Z} \to R \to S$ yields $\mathrm{D}_n(R|\mathbb{Z}; -) = 0$ for all $n \gg 0$. Theorem 2.1 now shows that $\mathrm{D}_n(R|\mathbb{Z}; -) = 0$ for $n \geq 2$ on the category of S-modules. Feeding this back into Remark 1.6 we obtain $\mathrm{D}_n(R|\mathbb{Z}; -) = 0$ for all $n \geq 3$. The argument for R is similar.

There exist homomorphisms with $\text{AQ-dim}_R\, S = 2$.

Example 2.5 Let $Q \xrightarrow{\psi} R \xrightarrow{\varphi} S$ be homomorphisms of noetherian rings such that $\varphi\psi$ is l.c.i. and ψ is c.i. at each prime ideal containing $\operatorname{Ker}(\varphi)$.

If φ is not l.c.i., then Theorem 1.3 and Remark 1.6 imply $\text{AQ-dim}_R S = 2$.

For instance, consider $R = Q/I$ and $S = Q/J$, where $I \subseteq J$ are ideals in Q generated by Q-regular sequences. The canonical maps ψ and φ define an exact sequence of S-modules

$$0 \longrightarrow \mathrm{D}_2(S\,|\,R;S) \longrightarrow \frac{I}{I^2+IJ} \longrightarrow \frac{J}{J^2} \longrightarrow \frac{J}{J^2+I} \longrightarrow 0$$

cf. Remark 1.6 and Theorem 1.3. From it one obtains an isomorphism

$$\mathrm{D}_2(S\,|\,R;S) \cong \frac{I \cap J^2}{I^2+IJ}\,.$$

Examples with $I \cap J^2 \neq (I^2+IJ)$ abound, cf. 4.5 for the simplest one.

The next result settles conjecture 2.2 for a new class of maps.

Theorem 2.6 *Let $Q \xrightarrow{\psi} R \xrightarrow{\varphi} S$ be homomorphisms of noetherian rings with $\varphi\psi$ l.c.i. The following conditions are equivalent.*

(i) *ψ is complete intersection at each $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \supseteq \operatorname{Ker}(\varphi)$.*

(ii) $\mathrm{D}_3(S\,|\,R;-) = 0$.

(iii) $\text{AQ-dim}_R S \leq 2$.

(iv) $\text{AQ-dim}_R S$ *is locally finite.*

If, in addition, S has characteristic 0, they are also equivalent to

(v) $\mathrm{D}_m(S\,|\,R;-) = 0$ *for some integer $m \geq 3$.*

Thanks to 1.6, the equivalence of the first three conditions is contained in Theorem 1.3. The proof that (iv) or (v) implies (i) is one of the main results of [13].

Remark 2.7 The theorem applies to ring homomorphisms $S \xrightarrow{\psi} R \xrightarrow{\varphi} S$ with $\varphi\psi = \mathrm{id}_S$.

Theorems 1.2 and 1.3 determine the structural properties of algebras of André-Quillen dimension ≤ 2. With those results in mind, the problem arises whether there is a structural characterization of algebras of finite André-Quillen dimension. As of now, the method described in 2.5 is essentially the only known way of generating such algebras, so one might wonder whether *all* of them arise in this way, at least locally. This raises the following

Problem 2.8 Let $\varphi\colon R \to S$ be a surjective homomorphism of complete local rings. If $\text{AQ-dim}_R S < \infty$, then does there exist a surjective local homomorphism $Q \to R$ such that the kernel of the composition $Q \to R \to S$ is generated by a regular sequence?

Theorem 2.6 and Remark 2.3 show that an affirmative answer to the problem above would imply the validity of Quillen's conjecture 2.2. That conjecture also focuses attention on algebras with $\text{AQ-dim}_R S = 2$. The following result of Blanco, Majadas, and Rodicio [15, Cor. 3′] translates this hypothesis into one involving more familiar homological notions.

Theorem 2.9 *Let $\varphi\colon R \to S$ be a surjective homomorphism of noetherian rings, and let E be the Koszul complex on a system of generators of the ideal* $\operatorname{Ker}(\varphi)$ *in R.*

The following conditions are equivalent.

(i) $\text{AQ-dim}_R S \le 2$.

(ii) *The S-module* $\mathrm{H}_1(E)$ *is projective and the canonical homomorphism of graded S-algebras* $\bigwedge_S \mathrm{H}_1(E) \to \mathrm{H}(E)$ *is bijective.*

Another open question on vanishing of André-Quillen homology functors is

Problem 2.10 Does $\mathrm{D}_m(S\,|\,R; -) = 0$ for some integer $m \ge 1$ imply $\text{AQ-dim}_R S < m$?

Theorems 2.1 and 2.6 provide evidence for a positive answer, at least when S has characteristic 0. More generally, the following hold when $\lfloor \frac{m+1}{2} \rfloor$ is invertible in S: if $\mathrm{fd}_R S$ is locally finite and $\mathrm{D}_m(S\,|\,R; -) = 0$, then $\text{AQ-dim}_R S \le 1$, cf. [9]; if $Q \to R \to S$ are homomorphisms of rings whose composition is l.c.i., and $\mathrm{D}_m(S\,|\,R; -) = 0$, then $\text{AQ-dim}_R S \le 2$, cf. [13].

Problem 2.10 illustrates how little is known about André-Quillen homology compared to classical homology—the functors $\mathrm{Tor}_n^R(S, -)$ on the category of R-modules. In the latter case, the well-known positive answer to the analogue of Problem 2.10 is the basis of the theory of flat dimension: vanishing of $\mathrm{Tor}_m^R(S, -)$ means that S has a resolution of length at most $(m-1)$ by flat R-modules, and so implies $\mathrm{Tor}_n^R(S, -)$ for all integers $n \ge m$.

The discussion above focuses on vanishing of André-Quillen homology *functors*. The significance of vanishing of homology with coefficients in the algebra itself is still unclear, although the following conjecture was made by Herzog [27] some twenty years ago.

Conjecture 2.11 A local algebra S, essentially of finite type over a field K of characteristic 0 and with $\mathrm{D}_n(S\,|\,K; S) = 0$ for all $n \ge 2$ (respectively, all $n \gg 0$), is complete intersection.

When S is in the linkage class of a complete intersection, the weak version holds, since then $\mathrm{D}_3(S\,|\,K; S) \ne 0 \ne \mathrm{D}_4(S\,|\,K; S)$ by a result of Ulrich [42]. The strong version holds when all finite S-modules have rational Poincaré series, cf. [27]: classes of local rings that have the latter property are described in cf. [7], but Anick [5] shows that not all do.

Not much more is known at present. By way of contrast, vanishing of Tor is now understood for a wide class of algebras. This is the contents of the next section.

3 Vanishing of Cartan-Eilenberg homology

In the (now rarely used, but precise) terminology of Cartan and Eilenberg [18], an S-algebra R with structure map $\psi\colon S \to R$ is said to be *supplemented* if there is a fixed homomorphism of rings $\varphi\colon R \to S$, called the *augmentation*, such that $\varphi\psi = \mathrm{id}_S$.

By definition, $\mathrm{Tor}_{\bullet}^R(S, S)$ is the *Cartan-Eilenberg homology* of the supplemented S-algebra R. There are easily established, well known isomorphisms

$$\mathrm{Tor}_0^R(S, S) \cong S$$
$$\mathrm{Tor}_1^R(S, S) \cong I/I^2 \quad \text{where} \quad I = \operatorname{Ker}\varphi\,.$$

The ⋔-product of Cartan-Eilenberg endows $\operatorname{Tor}^R_\bullet(S,S)$ with a structure of a graded-commutative S-algebra, cf. Lemma 3.4. It yields a canonical homomorphism of graded S-algebras

$$\lambda^\bullet_{S|R}\colon \textstyle\bigwedge^\bullet_S(I/I^2) \longrightarrow \operatorname{Tor}^R_\bullet(S,S)\,.$$

The following theorem, due to Rodicio [39], is proved at the end of this section.

Theorem 3.1 *Let R be a noetherian ring and a supplemented S-algebra, and let I denote the kernel of the augmentation. The following conditions are equivalent.*

(i) *The ideal I is locally generated by a regular sequence.*

(ii) *The S-module I/I^2 is projective and the map $\lambda^\bullet_{S|R}$ is bijective.*

(iii) *There exists an integer $m > 0$ such that $\operatorname{Tor}^R_n(S,S) = 0$ for all $n \geq m$.*

(iv) *There exist an even integer $i > 0$ and an odd integer $j > 0$ such that*

$$\operatorname{Tor}^R_i(S,S) = 0 = \operatorname{Tor}^R_j(S,S)\,.$$

The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) are elementary. The proof that (iv) implies (i) has two distinct parts. First, a projective resolution F of S over R is constructed such that $F \otimes_R S$ contains easily identifiable cycles in all even degrees or in all odd degrees; essentially the same argument, based on Tate's construction recalled in Remark 3.2 below, is used in [14], [17], and [39]. Second, the chosen cycles are shown to be non-homologous to zero; the arguments in [14] and [17] are specific to Hochschild homology, while Rodicio [39] applies a result of Avramov and Rahbar-Rochandel, on 'large' homomorphisms of local rings, cf. [31, (2.5)]. In the special case of supplemented algebras, in Lemma 3.6 we give a much simpler proof of that result, partly drawing on an idea of Herzog [26].

The proof of the theorem uses DG (=differential graded) algebras A that are *graded commutative*: with $|\ |$ denoting degrees, all homogeneous elements $a, b \in A$ satisfy

$$ab = (-1)^{|a||b|}ba\,, \quad \text{and} \quad a^2 = 0 \quad \text{when } |a| \text{ is odd}\,.$$

Remark 3.2 For each surjective homomorphism $R \to T$, Tate [41], cf. also [24, Ch. 1] or [8, §6], provides a factorization $R \to R\langle U\rangle \to T$, where $R\langle U\rangle$ is a DG algebra obtained from R by adjoining *exterior variables* in odd degrees and *divided powers variables* in even degrees, and the second map induces an isomorphism in homology. Any DG algebra $R\langle U\rangle$ with these properties is called a *Tate resolution* of T over R. Let U_{odd} and U_{even} denote the variables of U of odd and even degree respectively, and let $v^{(s)}$ be the sth divided power of a variable $v \in U_{\text{even}}$. As a graded R-module, $R\langle U\rangle$ is free and spanned by monomials

$$\left\{u_{i_1}\cdots u_{i_m}\cdot v_{j_1}^{(s_1)}\cdots v_{j_n}^{(s_n)} \;\middle|\; u_{i_g} \in U_{\text{odd}}\,,\ v_{j_h} \in U_{\text{even}}\,;\ m, n \geq 0\ ,\ s_h \geq 0\right\};$$

to get a basis, order the variables in U and pick the ordered monomials. Let $I^{(2)}(R\langle U\rangle)$ denote the R-submodule spanned by the monomials with $m + \sum_{h=1}^n s_h \geq 2$.

Remark 3.3 It is well known (but the proof is not entirely elementary, cf. [24, §1.7]) that the divided powers of the variables in U extend to a system of divided powers on the DG algebra $R\langle U\rangle$. This means that for *every* element a of even positive degree and each $s\geq 0$ an element $a^{(s)}$ of degree $s|a|$ is defined and satisfies a list of standard identities, among them

$$a^{(0)}=1\,,\qquad (s!)\cdot a^{(s)}=a^s \text{ for all } s\geq 1\,,\qquad (a+b)^{(s)}=\sum_{i=0}^{s}a^{(s-i)}\cdot b^{(i)}\,.$$

The differential of $R\langle U\rangle$ satisfies the Leibniz formula with divided powers:

$$\partial(a\cdot b)=\partial(a)\cdot b+(-1)^{|a|}a\cdot\partial(b)\quad\text{for all } a,b\,;$$
$$\partial\big(a^{(s)}\big)=\partial(a)\cdot a^{(s-1)}\quad\text{for all } a \text{ of even positive degree and all } s\geq 1\,.$$

When R is an algebra over a field of characteristic 0, the DG algebra $R\langle U\rangle$ can be obtained by adjoining *polynomial variables* in even degrees, and setting $v^{(s)}=v^s/(s!)$ for all $v\in U_{\text{even}}$.

Lemma 3.4 *The structure of algebra with divided powers on $R\langle U\rangle$ induces such a structure on $T\langle U\rangle=R\langle U\rangle\otimes_R T$, which passes it on to $\mathrm{H}_\bullet(T\langle U\rangle)=\mathrm{Tor}^R_\bullet(T,T)$. The divided powers on $\mathrm{Tor}^R_\bullet(T,T)$ are independent of the choice of Tate resolution used in their computation.*

Proof The passage from $R\langle U\rangle$ to $T\langle U\rangle$ is straightforward.

The Leibniz formula shows that in $T\langle U\rangle$ products of cycles are cycles, products of cycles and boundaries are boundaries, and divided powers of cycles are cycles. Let $z,z'\in T\langle U\rangle$ be cycles of even positive degree and $z'=z+\partial(y)$. Since y is the image of some $w\in R\langle U\rangle$ under the canonical surjection $\pi\colon R\langle U\rangle\to T\langle U\rangle$, we have $\partial(y)^{(i)}=\big(\pi\partial(w)\big)^{(i)}=\pi\big(\partial(w)^{(i)}\big)$ for all $i\geq 1$. Furthermore, $\partial\big(\partial(w)^{(i)}\big)=\partial^2(w)\cdot\partial(w)^{(i-1)}=0$. As $\mathrm{H}_n(R\langle U\rangle)=0$ for $n>0$, for each i there is an element $w_i\in R\langle U\rangle$ such that $\partial(w_i)=\partial(w)^{(i)}$. As a result,

$$z'^{(s)}=z^{(s)}+\sum_{i=1}^{s}z^{(s-i)}\cdot\partial(y)^{(i)}=z^{(s)}+\partial\bigg(\sum_{i=1}^{s}z^{(s-i)}\cdot\pi(w_i)\bigg)\,.$$

Thus, the structure of algebra with divided powers on $T\langle U\rangle$ descends to its homology.

If $R\langle V\rangle$ also is a Tate resolution of T over R, then it is easy to see, cf. [24, (1.8.6)], that there exists a morphism of DG algebras $R\langle V\rangle\to R\langle U\rangle$ over R that is compatible with the systems of divided powers. Such a map is necessarily a quasiisomorphism, hence so is the induced map $T\langle V\rangle\to T\langle U\rangle$. Consequently, the structures of algebra with divided powers on $\mathrm{Tor}^R_\bullet(T,T)$ does not depend on the choice of Tate resolution used for its definition.

Finally, recall that if F is any free resolution of T, then the the $\pitchfork$-product [18, Ch. XI] is the composition $\mathrm{H}(F\otimes_R T)\otimes_R\mathrm{H}(F\otimes_R T)\to\mathrm{H}((F\otimes_R F)\otimes_R T)\to\mathrm{H}(F\otimes_R T)$ of the Künneth homomorphism with a map induced by a morphism $F\otimes_R F\to F$ lifting the multiplication map $T\otimes_R T\to T$. Since the product map $R\langle U\rangle\otimes R\langle U\rangle\to R\langle U\rangle$ is such a morphism, the product structure above coincides with the classical one.

Remark 3.5 If $(R,\mathfrak{m},k)$ is a local ring and $R\to T$ is a surjective homomorphism, then in constructing a Tate resolution $R\langle U\rangle$ of T the following choices can be made: $\partial(U_1)$ minimally generates $\operatorname{Ker}\varphi$, and $\partial(U_n)$ minimally generates $\mathrm{H}_{n-1}(R\langle U_{<n}\rangle)$ for each $n\geq 2$.

Any resolution obtained in this way is called an *acyclic closure* of T over R. Acyclic closures are characterized among Tate resolutions of T over R by a property of their differential:

$$\partial(u) \in \mathfrak{m}U + I^{(2)}(R\langle U\rangle) \qquad \text{for all} \qquad u \in U\,.$$

If $R\langle X\rangle$ is an acyclic closure of k over R, then by Gulliksen [23] and Schoeller [40], cf. also [24, (1.6.4)] or [8, (6.3.4)], it is a *minimal* resolution of k over R, that is, $\partial(R\langle X\rangle) \subseteq \mathfrak{m}R\langle X\rangle$.

Lemma 3.6 *If $(R, \mathfrak{m}, k)$ is a local ring and R is a supplemented S-algebra, and $R\langle U\rangle$ is an acyclic closure of S over R, then $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$.*

Proof Noting that S is a local ring with residue field k, choose an acyclic closure $S\langle Y\rangle$ of k over S. Since $S\langle Y\rangle$ is a bounded below complex of free S-modules, and $R\langle U\rangle \to S$ is a quasiisomorphism, so is the first arrow in the composition

$$R\langle U,Y\rangle = R\langle U\rangle \otimes_S S\langle Y\rangle \to S \otimes_S S\langle Y\rangle = S\langle Y\rangle \to k\,.$$

The last arrow is a quasiisomorphism, so $R\langle U,Y\rangle$ is a Tate resolution of k over R. If $R\langle X\rangle$ is an acyclic closure of k over R, then the homology of both $k\langle U,Y\rangle = R\langle U,Y\rangle \otimes_R k$ and $k\langle X\rangle = R\langle X\rangle \otimes_R k$ compute $\mathrm{Tor}^R_\bullet(k,k)$. By Lemma 3.4 there are isomorphisms

$$\mathrm{H}_\bullet(k\langle U,Y\rangle) \cong \mathrm{H}_\bullet(k\langle X\rangle) = k\langle X\rangle$$

of algebras with divided powers: the equality holds by the theorem of Gulliksen and Schoeller, cf. Remark 3.5. The same theorem implies $\partial(Y) = 0$ in $k\langle U,Y\rangle$. Noting that $\partial(U_1) = 0$, assume by induction that $\partial(U_{\leqslant n}) = 0$ for some $n \geq 1$. Thus, the differential of the DG algebra with divided powers $k\langle U_{\leqslant n}, Y_{\leqslant n}\rangle$ is trivial. By the isomorphisms above, the inclusion $k\langle U_{\leqslant n}, Y_{\leqslant n}\rangle \hookrightarrow k\langle U,Y\rangle$ induces in homology an injective map, hence $\partial(U_{n+1}) = 0$.

□

Proof (of Theorem 3.1) The conditions in the theorem are local over the prime ideals $\mathfrak{p}$ of R that contain I. Localizing at such a prime we may assume that $(R, \mathfrak{m}, k)$ is a local ring. Fix a minimal set of generators $\boldsymbol{y} = y_1, \ldots, y_c$ for I and let E be the Koszul complex on $\boldsymbol{y}$.

(i) $\Longrightarrow$ (ii). Since $\boldsymbol{y}$ is a regular sequence, E is a Tate resolution of S over R, hence

$$\mathrm{Tor}^R_\bullet(S,S) = \mathrm{H}_\bullet(E \otimes_R S) = E \otimes_R S\,.$$

Thus, $\mathrm{Tor}^R_1(S,S)$ is equal to the free S-module $E_1 \otimes_R S$, and $\mathrm{Tor}^R_\bullet(S,S) = \bigwedge^\bullet_S(E_1 \otimes_R S)$.

(iv) $\Longrightarrow$ (i). Let $R\langle U\rangle$ be an acyclic closure of S over R, so $U_1 = \{u_1, \ldots, u_c\}$ with $\partial(u_h) = y_h$ for $h = 1, \ldots, c$, and $U_2 = \{v_1, \ldots, v_r\}$ with $\mathrm{cls}(\partial(v_1)), \ldots, \mathrm{cls}(\partial(v_r))$ minimally generating $\mathrm{H}_1(R\langle U_1\rangle)$. In $S\langle U\rangle = S \otimes_R R\langle U\rangle$ we have $\partial(U_1) = 0$ and $\partial(U_2) \subseteq SU_1$, hence

$$Z = S\langle u_1, \ldots, u_c\rangle \oplus S\langle v_1, \ldots, v_r\rangle\, u_1 \cdots u_c \subseteq S\langle U\rangle$$

is a submodule of cycles. By the lemma, $\partial(R\langle U\rangle) \subseteq \mathfrak{m}R\langle U\rangle$, so the composition

$$Z \otimes_S k \to \mathrm{H}_\bullet(S\langle U\rangle) \otimes_S k \to \mathrm{H}_\bullet(S\langle U\rangle \otimes_S k) = k\langle U\rangle$$

is injective. Thus, either $r = 0$, or $\mathrm{Tor}^R_n(S,S) = \mathrm{H}_n(S\langle U\rangle) \neq 0$ for all $n \in \mathbb{N}$ with $n \equiv c$ (mod 2). Our hypothesis bars the latter case, so $R\langle U_1\rangle \cong E$, hence the sequence $\boldsymbol{y}$ is regular.

□

4 Finite generation of Cartan-Eilenberg homology

Finite generation of the S-algebra $\mathrm{Tor}^R_\bullet(S,S)$ is described by the next result. A local structure theorem for such algebras is given in [13, (4.1.iii)].

Theorem 4.1 *Let R be a noetherian ring and a supplemented S-algebra.*

If the S-algebra $\mathrm{Tor}^R_\bullet(S,S)$ is finitely generated, then there exist projective S-modules D_i concentrated in degrees i for $i=1,2$, and an isomorphism of graded S-algebras

$$\textstyle\bigwedge_S D_1 \otimes_S \mathrm{Sym}_S D_2 \cong \mathrm{Tor}^R_\bullet(S,S)\,;$$

furthermore, $(D_2)_{\mathfrak{q}} = 0$ for each $\mathfrak{q} \in \mathrm{Spec}\, S$ with $k(\mathfrak{q}) > 0$.

Remark 4.2 The proof of the theorem can be reduced to the case of a local ring R.

Proof (of Theorem 4.1 in positive characteristic) Assume $(R,\mathfrak{m},k)$ is a local ring with $\mathrm{char}(k) = p > 0$. Let $a_1,\dots,a_s$ be a set of generators of positive degree of the S-algebra $\mathrm{Tor}^R_\bullet(S,S)$. Lemma 3.4 and Remark 3.3 show that if $|a_h|$ is even, then $(a_h)^p = p!(a_h)^{(p)}$, while if $|a_h|$ is odd, then $(a_h)^2 = 0$. Thus, $\mathrm{Tor}^R_n(S,S) = p\,\mathrm{Tor}^R_n(S,S)$ for all $n \gg 0$. By Nakayama's Lemma, $\mathrm{Tor}^R_n(S,S) = 0$ for all $n \gg 0$. It remains to invoke Theorem 3.1.

□

The proof is much more involved in characterisctic 0, so we restrict ourselves to describing its main steps. A crucial ingredient is provided by the following result.

Theorem 4.3 *Let R be a noetherian ring and a supplemented S-algebra.*

If the S-algebra $\mathrm{Tor}^R_\bullet(S,S)$ is finitely generated, then $\mathrm{D}_n(S\,|\,R;-) = 0$ for $n \geq 3$.

The proof of the last theorem itself has two distinct parts. The first is to show that finite generation of $\mathrm{Tor}^R_\bullet(S,S)$ implies $\mathrm{D}_n(S\,|\,R;-) = 0$ for $n \gg 0$. The argument occupies a good part of [12]. This theorem is related to a result of Dupont and Vigué-Poirrier [19] on rational cohomology of free loop spaces; the general principles of such a correspondence are explained in [11]. Once it is known that André-Quillen homology vanishes eventually, Theorem 2.6 shows that it vanishes from degree 3.

Using the preceding theorems one can compute all torsion and André-Quillen functors.

Corollary 4.4 *There are isomorphisms of functors on the category of S-modules:*

$$\mathrm{Tor}^R_n(S,-) \cong \left(\bigoplus_{i=0}^{\infty} \textstyle\bigwedge^i_S D_1 \otimes_S \mathrm{Sym}^{n-2i}_S D_2\right) \otimes_S - \quad \textit{for} \quad n \geq 0\,;$$

$$\mathrm{D}_n(S\,|\,R;-) \cong \begin{cases} D_n \otimes_S - & \textit{for} \quad n = 1,2\,; \\ 0 & \textit{otherwise}\,. \end{cases}$$

The converse to Theorem 4.3 need not hold:

Example 4.5 Let k be a field, $Q = k[[t,u]]$, $R = Q/(tu)$, and let $S = Q/(t)$. The canonical maps $k[[t]] \hookrightarrow k[[t,u]] \twoheadrightarrow k[[t]]$ induce on R the structure of a supplemented S-algebra.

By 2.5, one has $\mathrm{D}_n(S\,|\,R;-)=0$ for $n\geq 3$. Computing $\mathrm{Tor}^R_\bullet(S,S)$ from the free resolution $F_\bullet\ =\ \cdots \xrightarrow{u} R \xrightarrow{t} R \xrightarrow{u} R \to 0$, one obtains

$$\mathrm{Tor}^R_n(S,S)=\begin{cases} S & \text{if } n=0\\ k & \text{if } n\geq 1 \text{ and odd}\\ 0 & \text{otherwise.}\end{cases}$$

If $a,b\in \mathrm{Tor}^R_{\geqslant 1}(S,S)$ are non-zero elements, then ab has even degree, so $ab=0$. Thus, each set of algebra generators of $\mathrm{Tor}^R_\bullet(S,S)$ over S has an element in every positive odd degree.

While André-Quillen homology of algebras is constructed from simplicial resolution, in characteristic 0, it can be computed from DG algebras.

Remark 4.6 Let $R\to S$ be a surjective homomorphism of noetherian rings and let $R\langle U\rangle$ be a Tate resolution of S over R. When S has characteristic 0, Quillen [36, (9.5)] proves

$$\mathrm{D}_n(S\,|\,R;-)\cong \mathrm{H}_n(Q^\gamma(R\langle U\rangle)\otimes_R -)\quad \text{for}\quad n\in\mathbb{Z}\,,$$

where $Q^\gamma(R\langle U\rangle)=R\langle U\rangle\,/\big(R+I^{(2)}(R\langle U\rangle)\big)$, with $I^{(2)}(R\langle U\rangle)$ the graded R-module defined in Remark 3.2; by the Leibniz rule, $R+I^{(2)}(R\langle U\rangle)$ is a subcomplex of $R\langle U\rangle$, so that $Q^\gamma(R\langle U\rangle)$ is a complex of R-modules. Note that, as a graded R-module, $Q^\gamma(R\langle U\rangle)\cong RU$.

Lemma 4.7 *Let $R\to S$ be a surjective homomorphism of local rings such that S has characteristic* 0, *and let $R\langle U\rangle$ be an acyclic closure of S over R.*

For each integer n, the following conditions are equivalent:

(i) $U_n=\varnothing$.

(ii) $\mathrm{D}_n(S\,|\,R;-)=0$.

(iii) $\mathrm{D}_n(S\,|\,R;k)=0$.

Proof (i) $\implies$ (ii) is contained in the preceding remark.

(iii) $\implies$ (i). By construction of acyclic closures, cf. Remark 3.5, one has $\partial(U)\subseteq\mathfrak{m}R\langle U\rangle$, so that $\partial\big(Q^\gamma(R\langle U\rangle)\big)\subseteq\mathfrak{m}\,Q^\gamma(R\langle U\rangle)$, hence $Q^\gamma(R\langle U\rangle)\otimes_R k$ has trivial differential. Thus, by the last remark $kU_n\cong \mathrm{D}_n(S\,|\,R;k)=0$; that is to say, $U_n=\varnothing$.

□

The final step in the proof of Theorem 4.1 is abstracted from the argument for [12, (4.1)].

Proposition 4.8 *Let $(S,\mathfrak{n},k)$ be a local ring of charactersitic* 0 *and let $S\langle U_1,U_2\rangle$ be a DG algebra with U_i in degree i, $\partial(U_1)=0$ and $\partial(U_2)\subseteq\mathfrak{n}U_1$.*

If the S-algebra $\mathrm{H}(S\langle U_1,U_2\rangle)$ is finitely generated, then $\partial(U_2)=0$.

Proof Let $U_1=\{u_1,\dots,u_e\}$ and $U_2=\{v_1,\dots,v_c\}$; the products $u_{i_1}\cdots u_{i_g}\cdot v_1^{(s_1)}\cdots v_c^{(s_c)}$, with $i_1<\cdots<i_g$ and $s_h\geq 0$ form a basis of the graded S-module $A=S\langle U\rangle$. Assigning

to such a product *upper degree* $l = g + s_1 + \cdots + s_c$, we turn A into a bigraded DG algebra $A = \bigoplus_{0 \le l \le n} A_n^{(l)}$ with $\partial(A_n^{(l)}) \subseteq A_{n-1}^{(l)}$ and $H = \mathrm{H}_*(A)$ inherits the bigrading. Furthermore,

$$\begin{aligned} &H_0^{(0)} = A_0^{(0)} = S \\ &H_n^{(l)} = 0 && \text{for} \quad n < 2l - e \text{ or } n > 2l \\ &H_n^{(l)} \cdot H_{n'}^{(l')} \subseteq H_{n+n'}^{(l+l')} && \text{for all} \quad l, l', n, n' \end{aligned}$$

Therefore, one has a direct sum decomposition $H = C \oplus D$, where $C = \bigoplus_{n<2l} H_n^{(l)}$ is an ideal and $D = \bigoplus_n H_{2n}^{(n)}$ is a subalgebra. Properties (i) and (ii) show that $E = \bigoplus_n H_{2n+e}^{(n+e)}$ is an ideal of the graded algebra H and $CE = 0$. By hypothesis H is finitely generated as an algebra over the noetherian ring S, hence the ideal E of H is finitely generated. It follows that E is finite as a module over the algebra $H/C = D$.

The vanishing lines of $A_n^{(l)}$ yield exact sequences of graded S-modules

$$0 \to D \to S\langle U_2 \rangle \xrightarrow{\partial} S\langle U_2 \rangle \otimes_S SU_1 \,;$$
$$S\langle U_2 \rangle \otimes_S {\textstyle\bigwedge}^{e-1}(SU_1) \xrightarrow{\partial} S\langle U_2 \rangle \otimes_S {\textstyle\bigwedge}^{e}(SU_1) \to E \to 0 \,.$$

The map $b \in S\langle U_2 \rangle \mapsto b \cdot u_1 \cdots u_e \in S\langle U_2 \rangle \otimes_S \bigwedge^e(SU_1)$ is a degree c homomorphism $\tau \colon S\langle U_2 \rangle \to E$ of graded D-modules. As $\partial(S\langle U \rangle) \subseteq \mathfrak{n} S\langle U \rangle$, we see that

$$\tau \otimes_D k \colon S\langle U_2 \rangle \otimes_D k \to E \otimes_D k$$

is bijective. For each $n \in \mathbb{Z}$ the degree n component of the D-module $S\langle U_2 \rangle$ is a finite S-module, and vanishes for $n < 0$, so by the appropriate version of Nakayama's Lemma the D-module $S\langle U_2 \rangle$ is finite. In particular, each $v \in U_2$ satisfies an equation

$$v^s + z_{s-1} v^{s-1} + \cdots + z_1 v + z_0 = 0 \in S\langle U_2 \rangle$$

of integral dependence with $z_j \in D$. Differentiating one with minimal s, we get

$$\big(s v^{s-1} + (s-1) z_{s-1} v^{s-2} + \cdots + z_1 \big) \partial(v) = 0 \in S\langle U_2 \rangle \otimes_S SU_1 \,.$$

Since $\mathbb{Q} \subseteq S$, the minimality of s implies that the coefficient of $\partial(v)$ is non-zero, hence it is not a zero-divisor on the free $S\langle U_2 \rangle$-module $S\langle U_2 \rangle \otimes_S SU_1$, and so $\partial(v) = 0$.

□

Proof (of Theorem 4.1 in characteristic zero) By Remark 2.3, we assume $(R, \mathfrak{m}, k)$ is a local ring and a supplemented S-algebra. Since the S-algebra $\mathrm{Tor}^R_\bullet(S,S)$ is finitely generated, Proposition 4.3 yields $\mathrm{D}_n(S|R; -) = 0$ for $n \ge 3$. In view of Remark 4.6, this implies that S has an acyclic closure $R\langle U \rangle$ over R with $U = U_1 \sqcup U_2$, where U_i consists of variables of degree i. In the DG algebra $S\langle U \rangle = R\langle U \rangle \otimes_R S$, one has $\partial(U_1) = 0$ and $\partial(U_2) \subseteq \mathfrak{n} U_1$, where $\mathfrak{n}$ is the maximal ideal of S. Since $\mathrm{H}(S\langle U \rangle) = \mathrm{Tor}^R_\bullet(S,S)$ is finitely generated, the preceding proposition yields $\mathrm{Tor}^R_\bullet(S,S) = S\langle U \rangle = S\langle U_1 \rangle \otimes_S S\langle U_2 \rangle$. Finally, $S\langle U_1 \rangle = \bigwedge_S(SU_1)$ by defintion, while $S\langle U_2 \rangle = \mathrm{Sym}_S(SU_2)$ since S has characteristic 0.

□

5 Smoothness and Hochschild homology

Throughout this section K denotes a noetherian ring and S a flat K-algebra essentially of finite type. The ring $R = S \otimes_K S$ is then a noetherian supplemented S-algebra, with structure map $\psi\colon S \to R$ given by $s \mapsto s \otimes 1$, and augmentation $\varphi = \mu_K^S\colon S \otimes_K S \to S$.

By a classical result of Cartan and Eilenberg [18, Ch. IX], the Hochschild homology of the K-algebra S, as an algebra under shuffle products, can be computed as the homology algebra of the supplemented S-algebra R: There is an isomorphism of graded algebras

$$\mathrm{HH}_\bullet(S\,|\,K, S) \cong \mathrm{Tor}_\bullet^{S\otimes_K S}(S, S)\,.$$

In particular, this contains the basic equalities

$$\mathrm{HH}_0(S\,|\,K, S) = S \qquad \text{and} \qquad \mathrm{HH}_1(S\,|\,K, S) = \Omega_{S\,|\,K}\,.$$

The characterization of smoothness in terms of Hochschild homology, presented below, combines results proved over a period of 40 years.

Over a field, the implication (i) $\implies$ (ii) is a celebrated result of Hochschild, Kostant, and Rosenberg [28]. It was extended to noetherian rings by André [4], partly building on results of Quillen [36]. Since (ii) implies (iii), which in turn contains (iv) and (v), the fact that (i) follows from any of these properties represents a strong converse to the Hochschild-Kostant-Rosenberg Theorem.

When K is a field, Rodicio [37] conjectured that (iii) implies (i). The fact that (iv) implies (i) was proved by Avramov and Vigué-Poirrier [14]; independently, Campillo, Guccione, Guccione, Redondo, Solotar, and Villamayor [17] gave a proof when K is a field of characteristic 0. The general case, when K is a noetherian ring, was proved by Rodicio [39].

The implication (v) $\implies$ (i) is even more recent. When K is a field of characteristic 0 and S is positively graded with $S_0 = K$, it is proved by Dupont and Vigué-Poirrier in [44], [19]. The general case is the main result of our paper [12].

Theorem 5.1 *The following conditions are equivalent.*

(i) *The algebra S is smooth over K.*

(ii) *The S-module $\Omega_{S\,|\,K}$ is finite projective and* $\mathrm{HH}_\bullet(S\,|\,K, S) = \bigwedge^\bullet \Omega_{S\,|\,K}$.

(iii) *There exists an integer $m > 0$ such that $\mathrm{HH}_m(S\,|\,K) = 0$ for all $n \geq m$.*

(iv) *There exist an even integer $i > 0$ and an odd integer $j > 0$ such that*

$$\mathrm{HH}_i(S\,|\,K) = 0 = \mathrm{HH}_j(S\,|\,K)\,.$$

(v) *The S-algebra $\mathrm{HH}_\bullet(S\,|\,K)$ is finitely generated.*

Proof The equivalence of conditions (i) through (iv) is contained in Theorems 1.1 and 3.1. Since (ii) clearly implies (v), it remains to prove

(v) $\implies$ (i). The S-algebra $\mathrm{Tor}_\bullet^R(S, S) = \mathrm{HH}_\bullet(S\,|\,K)$ is finitely generated, so Theorem 4.1 applies. Fix a prime ideal $\mathfrak{q}$. If $\operatorname{char} k(\mathfrak{q}) = 0$, then the $S_\mathfrak{q}$-module $(\Omega_{S\,|\,K})_\mathfrak{q} \cong (D_1)_\mathfrak{q}$ is free, so $S_\mathfrak{q}$ is smooth over K by Theorem 1.1. If $\operatorname{char} k(\mathfrak{q}) > 0$, then $(D_2)_\mathfrak{q} = 0$, so the

$S_{\mathfrak{q}}$-module $\mathrm{HH}_1(S_{\mathfrak{q}}|K) = \mathrm{HH}_1(S|K)_{\mathfrak{q}}$ is free, and $\mathrm{HH}_\bullet(S_{\mathfrak{q}}|K) \cong \bigwedge_{S_{\mathfrak{q}}} \mathrm{HH}_1(S_{\mathfrak{q}}|K)$, so $S_{\mathfrak{q}}$ is smooth over K by the already established implication (ii) $\implies$ (i).

□

It is not possible, in general, to relax condition (iv) in Theorem 5.1 to the vanishing of a single Hochschild homology module. The relevant example is a particular instance of a general phenomenon discovered by Larsen and Lindenstrauss [30]: For any ring of algebraic integers $S \neq \mathbb{Z}$, one has $\mathrm{HH}_{2i-1}(S|\mathbb{Z}) \neq 0 = \mathrm{HH}_{2i}(S|\mathbb{Z})$ for all $i \geq 1$.

Example 5.2 The Hochschild homology of the $\mathbb{Z}$-algebra $S = \mathbb{Z}[\sqrt{3}]$ is given by

$$\mathrm{HH}_n(S|\mathbb{Z}) = \begin{cases} S & \text{for } n = 0; \\ S/(2\sqrt{3}) & \text{for odd } n \geq 1; \\ 0 & \text{for even } n \geq 2. \end{cases}$$

Since S is flat over $\mathbb{Z}$ and isomorphic to $\mathbb{Z}[x]/(x^2-3)$, the next result, or a direct computation, shows that the $\mathrm{HH}_\bullet(S|\mathbb{Z})$ is the homology of the complex

$$\cdots \xrightarrow{2x} S \xrightarrow{0} S \xrightarrow{2x} S \xrightarrow{0} S \to 0$$

concentrated in degrees 0 and higher. This yields the desired result.

Hochschild homology is somewhat better understood in the case of l.c.i. algebras, mostly due to the possibility of computing it from an elegant complex that can be derived by using Tate's procedure [41]. In characteristic 0 it was established by Wolffhardt [45]; in general, it appears in the papers of Guccione and Guccione [22] and Brüderle and Kunz [16].

Proposition 5.3 *Let $P = K[x_1, \ldots, x_e]$ be a polynomial ring over a noetherian ring K, and let I be an ideal such that $S = P/I$ is flat over K.*

If I is generated by a P-regular sequence $f_1, \ldots, f_c$, and $\partial_i(f_j)$ denotes the image in S of the partial derivative $\partial f_j/\partial x_i$, then $\mathrm{HH}_\bullet(S|K)$ is the homology of the DG algebra

$$S\langle U\rangle = S\Big\langle\, u_1, \ldots, u_e; v_1, \ldots, v_c \,\Big|\, \partial(u_i) = 0\,;\, \partial(v_j) = \sum_{i=1}^{e} \partial_i(f_j)u_i \,\Big\rangle\,.$$

This result has an interesting consequence. Bigrading $S\langle U\rangle$ as in Lemma 4.8, one obtains a decomposition of Hochschild homology modules: $\mathrm{HH}_n(S|K) = \bigoplus_{0\leq l\leq n} \mathrm{HH}_n^{(l)}(S|K)$. The strand $\mathrm{HH}_\bullet^{(1)}(S|K)$ is the homology of the complex of S-modules

$$0 \to \bigoplus_{j=1}^{c} Sv_j \xrightarrow{\left(\partial_i(f_j)\right)} \bigoplus_{i=1}^{e} Su_i \to 0\,,$$

concentrated in degrees 1 and 2. Since S is l.c.i. over K, this is a shift of the *cotangent complex* from which the André-Quillen homology of S over K is computed. Thus, the direct summand $\mathrm{HH}_n^{(1)}(S|K)$ of $\mathrm{HH}_n(S|K)$ is equal to $\mathrm{D}_{n-1}(S|K;S)$ for each n.

In characteristic 0, these patterns are found in the Hochschild homology of *all* flat K-algebras: this result is proved by Quillen [36, §8].

Theorem 5.4 *If* $\operatorname{char}(S) = 0$*, then the Hochschild homology modules admit decompositions such that the following properties hold for all integers* n, n'*.*

$$\mathrm{HH}_n(S|K) = \bigoplus_{0\le l\le n} \mathrm{HH}_n^{(l)}(S|K)\,;$$
$$\mathrm{HH}_n^{(1)}(S|K) \cong \mathrm{D}_{n-1}(S|K;S)\,;$$
$$\mathrm{HH}_n^{(l)}(S|K)\cdot \mathrm{HH}_{n'}^{(l')}(S|K) \subseteq \mathrm{HH}_{n+n'}^{(l+l')}(S|K)\,.$$

This structure of Hochschild homology is often referred to as its *Hodge decomposition* or λ-*decomposition*; it is known to arise in many different ways, cf. [33, 4.5].

The next question is suggested by inspection of examples of non-smooth algebras with a vanishing Hochschild homology module.

Problem 5.5 If $\mathrm{HH}_n(S|K) = 0$ for some $n > 0$, is then the ring $S_{\mathfrak q} \otimes_R k(\mathfrak q \cap K)$ a hypersurface for each prime ideal $\mathfrak q$ in S ?

Over a field, Rodicio [38] conjectures a rigid behavior:

Conjecture 5.6 If K is a field and $\mathrm{HH}_n(S|K)=0$ for some $n>0$, then S is smooth over K.

This has been settled in the affirmative in the following situations: by Rodicio [38] when S is a complete intersection, and by Vigué-Poirrier [43] when S is a non-negatively graded algebra with $S_0 = K$ a field of characteristic 0. The proof of the last result is interesting also because it brings a new homology theory into play: It characterizes smoothness of such algebras by the vanishing of a pair of *reduced cyclic homology* modules $\overline{\mathrm{HC}}_n(S|K)$ of different parity, compare Theorem 5.1. In characteristic 0, the cyclic homology of smooth algebras is completely determined by a theorem of Loday and Quillen, cf. [33, §3.4].

Further investigations of the links between properties of an algebra and properties of its cyclic homology might open new avenues of research.

References

[1] M. André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes Math. **32**, Springer-Verlag, Berlin, 1967.

[2] M. André, *Homologie des algèbres commutatives*, Grundlehren Math. Wiss. **206**, Springer-Verlag, Berlin, 1974.

[3] M. André, *Localisation de la lissité formelle*, manuscripta math. **13** (1974), 297–307.

[4] M. André, *Algèbres graduées associées et algèbres symétriques plates*, Comment. Math. Helv. **49** (1974), 277–301.

[5] D. Anick, *A counterexample to a conjecture of Serre*, Ann. of Math. (2) **115** (1982), 1–33; *Comment*, ibid. **116** (1982), 661.

[6] L. L. Avramov, *Flat morphisms of complete intersections*, [Russian] Dokl. Akad. Nauk. SSSR **225** (1975), 11–14; [translated in:] Soviet Math. Dokl. **16** (1975), 1413–1417.

[7] L. L. Avramov, *Local rings over which all modules have rational Poincaré series*, J. Pure Appl. Algebra **91** (1994), 29–48.

[8] L. L. Avramov, *Infinite free resolutions*, in: Six lectures in commutative algebra, Bellaterra, 1996, Progress in Math. **166**, Birkhäuser, Boston, 1998; pp. 1–118.

[9] L. L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*, Ann. of Math. (2) **150** (1999), 455-487.

[10] L. L. Avramov, H.-B. Foxby, B. Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994) 124–145.

[11] L. L. Avramov, S. Halperin, *Through the looking glass: A dictionary between rational homotopy theory and local algebra*, in: Algebra, algebraic topology, and their interactions; Stockholm, 1983, Lecture Notes Math. **1183**, Springer-Verlag, Berlin, 1986; pp. 1–27.

[12] L. L. Avramov, S. Iyengar, *Finite generation of Hochschild homology algebras*, Invent. Math. **140** (2000), 143–170.

[13] L. L. Avramov, S. Iyengar, *André-Quillen homology of algebra retracts*, Preprint, 2000.

[14] L. L. Avramov, M. Vigué-Poirrier, *Hochschild homology criteria for smoothness*, Internat. Math. Res. Notices **2** [in: Duke Math. J. **65** No. 2] (1992), 17–25.

[15] A. Blanco, J. Majadas, A. G. Rodicio, *On the acyclicity of the Tate complex*, J. Pure Appl. Algebra **131** (1998) 125–132.

[16] S. Brüderle, E. Kunz, *Divided powers and Hochschild homology of complete intersections*, Math. Ann. **299** (1994), 57–76.

[17] Buenos Aires Cyclic Homology Group, *A Hochschild homology criterium for the smoothness of an algebra*, Comment. Math. Helv. **69** (1994), 163–168.

[18] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1956.

[19] M. Dupont, M. Vigué-Poirrier, *Finiteness conditions for Hochschild homology algebra and free loop space cohomology algebra*, K-Theory, to appear.

[20] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Graduate Texts in Math. **150**, Springer-Verlag, Berlin, 1994.

[21] A. Grothendieck, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. I.H.E.S. **20**, **24**, **28**, **32**, (1967).

[22] J. A. Guccione, J. J. Guccione, *Hochschild homology of complete intersections*, J. Pure Appl. Algebra **74** (1991), 159–176.

[23] T. H. Gulliksen, *A proof of the existence of minimal algebra resolutions*, Acta Math. **120** (1968), 53–58.

[24] T. H. Gulliksen, G. Levin, *Homology of local rings*, Queen's Papers Pure Appl. Math. **20**, Queen's Univ., Kingston, ON, 1969.

[25] D. Harrison, *Commutative algebra and cohomology*, Trans. Amer. Math. Soc. **104** (1962), 191–204.

[26] J. Herzog, *Algebra retracts and Poincaré series*, manuscripta math. **21** (1977), 307–314.

[27] J. Herzog, *Homological properties of the module of differentials*, Atas da 6^a Escola de Álgebra (Recife), Coleç. Atas **14**, Soc. Brasil. Mat., Rio de Janeiro, 1981; pp. 35–64.

[28] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on a regular affine algebra*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[29] E. Kunz, *Kähler differentials*, Adv. Lectures Math. **4**, Vieweg, Braunschweig, 1986.

[30] M. Larsen, A. Lindenstrauss, *Cyclic homology of Dedekind domains*, K-Theory **6** (1992), 301–334.

[31] G. Levin, *Large homomorphisms of local rings*, Math. Scand. **46** (1980), 209–215.

[32] S. Lichtenbaum, M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. **128** (1967), 41–70.

[33] J.-L. Loday, *Cyclic homology*, Grundlehren Math. Wiss. **301**, Springer-Verlag, Berlin, 1992.

[34] H. Matsumura, *Commutative ring theory*, Stud. Adv. Math. **8**, Univ. Press, Cambridge, 1986.

[35] D. Quillen, *Homotopical algebra*, Lecture Notes Math. **43**, Springer-Verlag, Berlin, 1967.

[36] D. Quillen, *On the (co-)homology of commutative rings*, in: Applications of categorical algebra; New York, 1968, Proc. Symp. Pure Math. **17**, Amer. Math. Soc., Providence, RI, 1970; pp. 65–87.

[37] A. G. Rodicio, *Smooth algebras and vanishing of Hochschild homology*, Commentarii Math. Helv. **65** (1990), 474–477.

[38] A. G. Rodicio, *On the rigidity of the generalized Koszul complexes with applications to Hochschild homology*, J. Algebra **167** (1994), 343–347.

[39] A. G. Rodicio, *Commutative augmented algebras with two vanishing homology modules*, Adv. Math. **111** (1995), 162–165.

[40] C. Schoeller, *Homologie des anneaux locaux noethériens*, C. R. Acad. Sci. Paris Sér. A **265** (1967), 768–771.

[41] J. Tate, *Homology of noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–25.

[42] B. Ulrich, *Vanishing of cotangent functors*, Math. Z. **196** (1987), 463–484.

[43] M. Vigué-Poirrier, *Critères de nullité pour l'homologie des algèbres graduées*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 647–649.

[44] M. Vigué-Poirrier, *Finiteness conditions for the Hochschild homology algebra of a commutative algebra*, J. Algebra **207** (1998), 333–341.

[45] K. Wolffhardt, *The Hochschild homology of complete intersections*, Trans. Amer. Math. Soc. **171** (1972), 51–66.