STRUCTURE OF LOCAL HOMOMORPHISMS

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Introduction

A by now standard and most productive technique for studying local properties of commutative noetherian rings proceeds in two stages: The corresponding problem is first treated in the maximal–ideal–adic completions of a ring, and then the flatness of completion is used in order to descend the information to the initial ring.

The success of the first step largely depends on a fundamental result of commutative algebra – I. S. Cohen's Structure Theorem which shows that a *complete* local ring may be presented as a homomorphic image of a ring of formal power series over a complete discrete valuation domain; thus, questions on complete local rings are reduced to questions on ideals in specific rings.

The second step has been developed by A. Grothendieck into a full-fledged and versatile theory of flat descent. The idea behind it is that if $\varphi : R \to S$ is a *faithfully flat* homomorphism of commutative rings, then the properties of R and of the fibers of φ determine and are determined by the properties of S; this is in accordance with geometric intuition, which perceives such homomorphisms as algebraic substitutes for fiber bundles.

The purpose of this paper is to introduce the *local stage* of a new approach to the study of arbitrary homomorphisms of noetherian rings. If $\varphi : R \to S$ is a local homomorphism, it involves breaking down the canonically associated homomorphism $\grave{\varphi}$ from R to the *completion* of S into a composition of a flat homomorphism followed by a surjective one. The flat map has a regular closed fiber, hence well known results on flat descent and ascent can be applied. The surjective homomorphism can be investigated by means of powerful methods of ideal theory.

The preceding part of our procedure plays a role similar to that of Cohen presentations in the study of complete local rings. At a second stage we have to solve a descent problem for homomorphisms, namely, to transfer information from $\grave{\varphi}$ to φ. This requires a whole new bag of tricks (mostly – homological ones), and will be dealt with in an upcoming series of publications.

The layout of the present paper is as follows. In Section 1 we prove the existence of the factorizations mentioned above, by using Cohen's theorem and Grothendieck's theory of

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formal smoothness; we also discuss the uniqueness of the factorizations. The results are used in Section 2 to define analogues for local homomorphisms of some basic invariants of local rings – dimension, depth, and Cohen–Macaulay defect; these quantities are then compared to the corresponding ones for rings, but a deeper investigation is relegated to [4]. In Section 3 we focus on homomorphisms of *finite flat dimension*, for which there are particularly tight relations between invariants of the rings and of the homomorphism. We also introduce and briefly discuss a class of local homomorphisms which provides a natural extension of the class of Cohen–Macaulay local rings.

Part of the material of the last two sections was initally obtained in [4], by a completely different approach. The constructions of the first section allow for a more direct introduction of the basic concepts, as well as for simplifications of some arguments. They also yield definitions of other invariants, like embedding dimension, complete intersection defect, Cohen–Macaulay type, etc.; these are deferred to later publications, as their treatment requires different sets of techniques.

For the rest of the paper $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ denotes a *local homomorphism*, that is, a homomorphism of commutative noetherian rings, which maps the unique maximal ideal $\mathfrak{m}$ of R into the unique maximal ideal $\mathfrak{n}$ of S; whenever convenient, we shall simply write "$\varphi : R \to S$ is a local homomorphism".

1. Cohen Factorizations

A local homorphism $\tau : (R, \mathfrak{m}) \to (T, \mathfrak{p})$ is said to be *weakly regular* if it is flat andf its closed fiber $T/\mathfrak{m}T$ is a regular ring. Following [5, (4.1)], we say that a local homorphism $\varphi : R \to S$ is *factorizable* if it can be decomposed as $\varphi = \sigma\tau$ with $\tau : R \to T$ weakly regular and $\sigma : T \to S$ surjective; often we shall refer to such a situation by saying that $\varphi = \sigma\tau$ is a *regular factorization* of φ. Here are two basic instances.

(1.0.1) **Example.** A local homomorphism $\varphi : R \to S$ is said to be *essentially of finite type* if it can be factored as

$$R \xrightarrow{\tau} R[X]_{\mathfrak{M}} = T \xrightarrow{\sigma} S$$

where $R[X] = R[X_1, \ldots, X_n]$ is a polynomial ring, $\mathfrak{M}$ is a prime ideal in $R[X]$ lying over $\mathfrak{m}$, the homomorphism $\tau : R \to T$ is the localization map, and the homomorphism σ is surjective.

Any such factorization yields a regular factorization of φ.

(1.0.2) **Example.** Let $p \geq 0$ be the characteristic of the residue field $S/\mathfrak{n}$ of the local ring S. The homomorphism $\eta_S : \mathbb{Z}_{(p)} \to S$, obtained by localization from the canonical homomorphism $\mathbb{Z} \to S$, will be called the *local structure homomorphism* of S.

Cohen's Structure Theorem, cf. [7, Theorem 12] or [9, (19.8.8.1)], asserts that if S is complete, then there is a complete discrete valuation domain D with maximal ideal pD, and residue field $D/pD = S/\mathfrak{n}$, and a local homomorphism $D \to S$ which induces the identity on the residue fields; such a ring and homomorphism are called, respectively, a *Cohen ring* and a *coefficient homomorphism* for S. Furthermore, if $y_1, \ldots, y_n$ is a set of generators for $\mathfrak{n}$, and $D[[Y]] = D[[Y_1, \ldots, Y_n]]$ is a formal power series ring, then any coefficient homomorphism has a unique extension to a surjective homomorphism of rings

$D[[Y]] \to S$ which sends Y_j to y_j for $j = 1, \dots, n$. Thus, the local structure homomorphism η_S has a regular factorization

$$\mathbb{Z}_{(p)} \xrightarrow{\eta_{D[[Y]]}} D[[Y]] = T \xrightarrow{\sigma} S .$$

We let $\widehat{R}$ and $\widehat{S}$ denote the $\mathfrak{m}$–adic and $\mathfrak{n}$–adic completions of R and S, respectively. A local homomorphism $\varphi : R \to S$ induces by continuity a map $\widehat{\varphi} : \widehat{R} \to \widehat{S}$, which is a local homomorphism, called the *completion* of φ. We shall also consider the *semi-completion* of φ, that is, the local homomorphism $\grave{\varphi} : R \to \widehat{S}$ obtained by composing the canonical inclusion $R \to \widehat{R}$ with $\widehat{\varphi}$ or – equivalently – by composing φ with the canonical inclusion $S \to \widehat{S}$.

A regular factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ of φ with R' a complete local ring will be called a *Cohen factorization.* Clearly, in order for φ to admit a Cohen factorization, the local ring S has to be complete. Cohen's theorem shows that the local structure homomorphism of such a ring admits a Cohen factorization. Our first result stretches this statement to cover any local homomorhism to a complete ring.

(1.1) **Theorem (Existence of Cohen Factorizations).** *For any local homomorphism $\varphi : R \to S$ there is a Cohen factorization*

$$\begin{array}{ccccc} & & R' & & \\ & \dot{\varphi} \nearrow & & \searrow \varphi' & \\ R & & \xrightarrow[\grave{\varphi}]{} & & \widehat{S} \end{array} .$$

(1.1.1) *Remark.* Under the additional assumption that the induced residue field extension $R/\mathfrak{m} \hookrightarrow S/\mathfrak{n}$ is *separable*, the result is noted in [5, (4.2.2)]. The proof given below expands on that of [11, (1.6)].

(1.1.2) *Remark.* Recall that a local homomorphism $\tau : (R, \mathfrak{m}) \to (T, \mathfrak{p})$ is called *formally smooth* (or: $\mathfrak{p}$*–smooth*) if it is flat, and the closed fiber $T/\mathfrak{m}T$ is a *geometrically* regular algebra over the residue field of R. Clearly, formally smooth homomorphisms are weakly regular; the converse also holds when the induced residue field extension is separable, cf. [9, (19.6.4)] or [13, (Lemma 1, p. 216)], but fails in general.

We say that a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is *formally smoothable*, if it admits a decomposition $\varphi = \sigma\tau$ with τ formally smooth and σ surjective. Thus, a formally smoothable homomorphism is factorizable, and the converse holds when the residue field extension $R/\mathfrak{m} = k \hookrightarrow \ell = S/\mathfrak{n}$ is separable. The precise condition for a local homomorphism to a complete local ring to be formally smoothable involves the Cartier imperfection module $\Gamma_{\ell|k}$, cf. [9, (20.6.1)] or [13, p. 205], which can also be defined as the homology group $\mathrm{H}_1(k, \ell, \ell)$ in the André-Quillen homology theory of commutative algebras, cf. [1]: It is shown in [8] that $\grave{\varphi} : R \to \widehat{S}$ is formally smoothable if and only if $\dim_\ell \Gamma_{\ell|k} < \infty$.

Proof of (1.1). If $\widehat{\varphi} : \widehat{R} \to \widehat{S}$ has a Cohen factorization $\widehat{\varphi} = \widehat{\varphi}'\dot{\widehat{\varphi}}$, and $\kappa : R \to \widehat{R}$ is the completion map, then it is easy to see that $\varphi' = \widehat{\varphi}'$ and $\dot{\varphi} = \dot{\widehat{\varphi}}\kappa$ provide a Cohen factorization for $\dot{\varphi} = \widehat{\varphi}\kappa$. Thus, it suffices to prove the theorem assuming that both R and S are complete.

Consider then the commutative diagram of local homomorphisms

$$\begin{array}{ccccc} \mathbb{Z}_{(p)} & & \xrightarrow{\eta_{D[[Y]]}} & & D[[Y]] \\ \eta_C \downarrow & & & & \downarrow \sigma' \\ C & \underset{\rho'}{\to} & R & \underset{\varphi}{\to} & S \end{array}$$

in which C is a Cohen ring for R, D is a Cohen ring for S, while ρ' and σ' are, respectively, a coefficient homomorphism and a surjective homomorphism given by Cohen's Structure Theorem, recalled in (1.0.2). As C is a Cohen ring, the ring $D[[Y]]$ is complete, and the homomorphism σ' is surjective, there exists a local homomorphism $\xi' : C \to D[[Y]]$ which preserves the commutativity of the diagram, cf. [9, (19.8.6.i)].

Let now $x_1, \dots, x_m$ be a set of generators of the maximal ideal $\mathfrak{m}$ of R, and let ρ be the extension of the coefficient homomorphism ρ' to a ring homomorphism $C[[X]] = C[[X_1, \dots, X_m]] \to R$ defined by $\rho(X_i) = x_i$ for $i = 1, \dots, m$. Furthermore, extend σ' to a homomorphism $\sigma : D[[X, Y]] \to S$ by setting $\sigma(X_i) = \varphi(x_i)$, and extend ξ' to a local homomorphism $\xi : C[[X]] \to D[[X, Y]]$ by setting $\xi(X_i) = X_i$, for $i = 1, \dots, m$. By construction, the square

$$\begin{array}{ccc} C[[X]] & \xrightarrow{\xi} & D[[X,Y]] \\ \rho \downarrow & & \downarrow \sigma \\ R & \underset{\varphi}{\longrightarrow} & S \end{array}$$

is commutative, with surjective vertical arrows. Note that ξ is flat: This is a consequence of the flatness of ξ', which is straightforward, since C is a discrete valuation ring with maximal ideal generated by p, and $D[[Y]]$ has no p–torsion.

Set $R' = R \otimes_{C[[X]]} D[[X, Y]]$, and let $\dot{\varphi} : R \to R'$ and $\varphi' : R' \to S$ be the canonical homomorphisms defined by $\dot{\varphi}(r) = r \otimes 1$ and $\varphi'(r \otimes u) = \varphi(r)\sigma(u)$. As the equality $\varphi = \varphi'\dot{\varphi}$ is clear, it remains to show that it yields a Cohen factorization of φ.

First, the surjectivity of φ' is implied by that of σ.

Next, as $\dot{\varphi} = R \otimes_{C[[X]]} \xi$, the flatness of $\dot{\varphi}$ follows from that of ξ.

Furthermore, the fiber of $\dot{\varphi}$ is a regular local ring, by the following computation:

$$\begin{aligned} R'/\mathfrak{m}R' &\cong (R/\mathfrak{m}) \otimes_R (R \otimes_{C[[X]]} D[[X, Y]]) \\ &\cong (C[[X]]/(p, X)C[[X]]) \otimes_{C[[X]]} D[[X, Y]] \\ &\cong D[[X, Y]]/(p, X)D[[X, Y]] \\ &\cong (D/pD)[[Y]] \ . \end{aligned}$$

Finally, the local ring R' is complete, due to the canonical isomorphism:

$$R' \cong D[[X,Y]]/(\operatorname{Ker}\rho)D[[X,Y]] \ .$$

□

The proof of the preceding theorem shows that a homomorphism to a complete local ring admits various Cohen factorizations. The next result establishes that any two such factorizations are linked in a particularly simple manner to a third one.

In order to relate two Cohen factorizations $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ and $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} \widehat{S}$ of $\dot\varphi : R \to \widehat{S}$ we introduce some more terminology. A local homomorphism $v : R' \to R_1$ is said to be a *comparison* of $\varphi'\dot\varphi$ to $\varphi_1'\dot\varphi_1$ when the diagram

$$\begin{array}{ccccc} & & R' & & \\ & \dot\varphi \nearrow & v \downarrow & \searrow \varphi' & \\ R & \underset{\dot\varphi_1}{\longrightarrow} & R_1 & \underset{\varphi_1'}{\longrightarrow} & \widehat{S} \end{array}$$

is commutative. If v is surjective, we say that $\varphi_1'\dot\varphi_1$ is a *reduction* of $\varphi'\dot\varphi$, and also that $\varphi'\dot\varphi$ is a *deformation* of $\varphi_1'\dot\varphi_1$.

(1.2) **Theorem (Comparison of Cohen Factorizations).** *Any two Cohen factorizations $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} \widehat{S}$ and $R \xrightarrow{\dot\varphi_2} R_2 \xrightarrow{\varphi_2'} \widehat{S}$ of a local homomorphism $\dot\varphi : R \to \widehat{S}$ have a common deformation.*

If v is a deformation of $\varphi_1'\dot\varphi_1$ to $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$, then Ker v *is generated by an R'–regular sequence, whose elements are linearly independent over $R'/\mathfrak{m}'$ in $\mathfrak{m}'/(\mathfrak{m}'^2 + \mathfrak{m}R')$, where $\mathfrak{m}'$ is the maximal ideal of R'.*

Remark. It will be shown below in (1.7) and (1.8) that under additional assumptions there might be a *direct* comparison of $\varphi_1'\dot\varphi_1$ to $\varphi_2'\dot\varphi_2$, but that this will not be the case in general.

In the proof which follows, the use of fiber products is inspired by Grothendieck's proof in [10, (19.3.2)] of the independence of the property of a local ring S to be a complete intersection from the presentation of $\widehat{S}$ as a quotient of a regular local ring.

Proof. Consider the fiber product

$$Q = \{(r_1, r_2) \in R_1 \times R_2 \mid \varphi_1'(r_1) = \varphi_2'(r_2)\} \ .$$

This is a noetherian complete local ring: cf. e.g. [10, (19.3.2.1)]. The natural projections $R_1 \leftarrow R_1 \times R_2 \rightarrow R_2$ induce surjective homomorphisms

$$R_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_2} R_2$$

such that $\varphi_1'\pi_1 = \varphi_2'\pi_2$. Let $R \overset{\dot\varphi}{\to} P \overset{\upsilon}{\to} Q$ be a Cohen factorization of the local homomorphism $R \to Q$, $r \mapsto (\dot\varphi_1(r), \dot\varphi_2(r))$, and set:

$$\begin{aligned} \upsilon_1 = \pi_1\upsilon \quad &: P \to R_1 \ ; \\ \upsilon_2 = \pi_2\upsilon \quad &: P \to R_2 \ ; \\ \varphi' = \varphi_1'\upsilon_1 = \varphi_2'\upsilon_2 &: P \to \widehat{S} \ . \end{aligned}$$

It is easy to see that $\dot\varphi = \varphi'\dot\varphi$ is a Cohen factorization, and that υ_1 and υ_2 are reductions to $\varphi_1'\dot\varphi_1$ and $\varphi_2'\dot\varphi_2$, respectively. Thus the first assertion of the theorem has been established.

Let $a_1, \ldots, a_r$ be a minimal system of generators of $\operatorname{Ker}\upsilon$. Due to the R–flatness of R', the exact sequence

$$0 \longrightarrow \operatorname{Ker}\upsilon \longrightarrow R' \overset{\upsilon}{\longrightarrow} R_1 \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \operatorname{Ker}\upsilon/\mathfrak{m}(\operatorname{Ker}\upsilon) \longrightarrow R'/\mathfrak{m}R' \overset{\overline{\upsilon}}{\longrightarrow} R_1/\mathfrak{m}R_1 \longrightarrow 0 \ ,$$

which shows that $\overline{a}_1 = a_1 + \mathfrak{m}R', \ldots, \overline{a}_r = a_r + \mathfrak{m}R'$ is a minimal system of generators of $\operatorname{Ker}\overline{\upsilon}$. As by assumption both rings $R'/\mathfrak{m}R'$ and $R_1/\mathfrak{m}R_1$ are regular, and the homomorphism $\overline{\upsilon}$ is surjective, a well known result, cf. e.g. [13, (14.2)], shows that the sequence $\overline{a}_1, \ldots, \overline{a}_r$ is $R'/\mathfrak{m}R'$–regular, and its elements are linearly independent over $R'/\mathfrak{m}'$ modulo $(\mathfrak{m}'/\mathfrak{m}R')^2$.

The latter condition means that $a_1, \ldots, a_r$ are linearly independent modulo $({\mathfrak{m}'}^2 + \mathfrak{m}R')$.

Due to the flatness of $\dot\varphi$, the former condition implies that the sequence $a_1, \ldots, a_r$ is R'–regular, by an application of the following well known result. □

(1.3) **Lemma** [13, (22.5.Corollary)]. *Let $(R, \mathfrak{m}) \to (R,, \mathfrak{m}')$ be a flat local homomorphism.*

The images in $R'/\mathfrak{m}R'$ of a sequence $a_1, \ldots, a_r$ of elements in $\mathfrak{m}'$ then form an $R'/\mathfrak{m}R'$–regular sequence if and only if the sequence $a_1, \ldots, a_r$ is regular in R' and the induced homomorphism $R \to R'/(a_1, \ldots, a_r)R'$ is flat. □

Next we turn to the problem of improving a given Cohen factorization. Our strategy is based on an elementary remark, in which $\operatorname{edim} R$ denotes the *embedding dimension* of R, that is, the minimal number of generators of its maximal ideal $\mathfrak{m}$.

If $R \overset{\dot\varphi}{\to} R' \overset{\varphi'}{\to} \widehat{S}$ is a Cohen factorization of the local homomorphism $\dot\varphi : R \to \widehat{S}$, then there is an inequality:

$$\dim R' - \dim R \geq \operatorname{edim}(S/\mathfrak{m}S) \ . \tag{1.4}$$

Indeed, the surjection $\varphi' : R' \to \widehat{S}$ induces a surjective homomorphism:

$$\overline{\varphi'} : R'/\mathfrak{m}R' \to \widehat{S}/\mathfrak{m}\widehat{S}$$

which implies that $\operatorname{edim}(R'/\mathfrak{m}R') \geq \operatorname{edim}(\widehat{S}/\mathfrak{m}\widehat{S})$; as the embedding dimension is invariant under completion, the desired inequality follows from the equalities $\operatorname{edim}(R'/\mathfrak{m}R') = \dim(R'/\mathfrak{m}R') = \dim R' - \dim R$, due to the regularity of $R'/\mathfrak{m}R'$ and to the flatness of $\dot{\varphi}$, respectively.

A Cohen factorization is said to be *minimal*, if equality holds in (1.4) above. Note that the formula $\operatorname{Ker} \overline{\varphi'} = (\operatorname{Ker} \varphi')(R'/\mathfrak{m}R')$ shows that a Cohen factorization is minimal if and only if $\operatorname{Ker} \varphi' \not\subseteq \mathfrak{m}'^2 + \mathfrak{m}R'$.

(1.5) **Proposition.** *Any Cohen factorization has a reduction to a minimal one.*

A minimal Cohen factorization has no proper reduction, that is, if υ is a reduction of a minimal Cohen factorization, then it is an isomorphism.

Proof. Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization with $\dim R' - \dim R > \operatorname{edim}(S/\mathfrak{m}S)$. By the remarks above there is then an $x \in (\operatorname{Ker} \varphi) \setminus (\mathfrak{m}'^2 + \mathfrak{m}R')$. As the image $\overline{x}$ of x in $R'/\mathfrak{m}R'$ is different from 0, we conclude from (1.3) that x is a non zero divisor on R', and that the composition $\ddot{\varphi}$ of $\dot{\varphi}$ with $\upsilon : R' \to R'/(x)$ is flat. Remark next that the fiber of $\ddot{\varphi}$ is canonically isomorphic to the factor ring of the regular ring $(R'/\mathfrak{m}R')$ modulo $\overline{x}$, and hence is regular, as $\overline{x}$ is not in the square of the maximal ideal. Thus, the composition $R \xrightarrow{\ddot{\varphi}} R'/(x) \xrightarrow{\varphi''} \widehat{S}$ yields a Cohen factorization.

This factorization is obtained as a reduction of the original one, and $\dim(R'/(x)) - \dim R - \operatorname{edim}(S/\mathfrak{m}S)$ is smaller than $\dim R' - \dim R - \operatorname{edim}(S/\mathfrak{m}S)$. It follows that $\varphi'\dot{\varphi}$ reduces to a minimal Cohen factorization after a finite number of iterations of the preceding construction.

Let now υ be a reduction of a minimal Cohen factorization, $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$. On the one hand, we know from (1.2) that the kernel of the surjective homomorphism υ is generated by elements which are linearly independent in $\mathfrak{m}'/(\mathfrak{m}'^2 + \mathfrak{m}R')$. On the other hand, in view of the remarks made before the statement of the theorem, the inclusion $\operatorname{Ker} \upsilon$ is contained in $(\mathfrak{m}'^2 + \mathfrak{m}R')$. This is only possible if $\operatorname{Ker} \upsilon = 0$. □

As an application of the last result we establish a lifting theorem for weakly regular homomorphisms.

(1.6) **Theorem.** *Consider a surjective local homomorphism $\pi : P \to R$ and a weakly regular local homomorphism $\dot{\varphi} : R \to R'$.*

If the local ring R' is complete, then there exists a commmutative diagram of local homomorphisms:

$$\begin{array}{ccc} P & \xrightarrow{\dot{\zeta}} & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ R & \xrightarrow[\dot{\varphi}]{} & R' \end{array}$$

where P' is complete, $\dot{\zeta}$ is weakly regular, π' is surjective, and the induced map $\upsilon : R \otimes_P P' \to R'$ is an isomorphism.

Proof. Let $P \xrightarrow{\pi'} P' \xrightarrow{\dot\zeta} R'$ be a minimal Cohen factorization of $\dot\varphi\pi$.

Clearly, the homomorphism $\upsilon : R \otimes_P P' \to R'$ induced by the universal property of the tensor product is surjective. Furthermore, the homomorphism $R \otimes \dot\zeta : R \to R \otimes_P P'$ is flat, and its fiber $k \otimes_R (R \otimes_P P')$ is canonically isomorphic to the regular ring $k \otimes_P P'$. Thus, $\upsilon \circ (R \otimes \dot\zeta)$ is a Cohen factorization of $\dot\varphi$, and the commutative diagram:

$$\begin{array}{ccccc} & & R \otimes_P P' & & \\ & {\scriptstyle R\otimes\dot\zeta}\nearrow & \upsilon \downarrow & \searrow \upsilon & \\ R & \underset{\dot\varphi}{\longrightarrow} & R' & \underset{id_{R'}}{\longrightarrow} & R' \end{array}$$

shows that υ is a reduction of $\upsilon \circ (R \otimes \dot\zeta)$ to $id_{R'} \circ \dot\varphi$. By the preceding theorem, the fact that υ is bijective will follow once we show that the Cohen factorization $\upsilon \circ (R \otimes \dot\zeta)$ is minimal.

To see this, note that $\dim(R \otimes_P P') - \dim R = \dim P' - \dim P$ by the canonical isomorphism above, that the latter difference is equal to $\dim(P'/\mathfrak{m}P') = \operatorname{edim}(P'/\mathfrak{m}P')$ as $\dot\zeta$ is weakly regular, and finally that $\operatorname{edim}(P'/\mathfrak{m}P') = \operatorname{edim}(R'/\mathfrak{m}R')$ by the minimality of the Cohen factorization $\pi'\dot\zeta$. □

The statement obtained from the preceding result by replacing "weakly regular" by "formally smooth" is also true, and forms the existence part of [9, (19.7.2)]; under formally smooth assumptions, [*loc. cit.*] establishes also that a lifting such as $\dot\zeta$ is essentially unique.

For a local homomorphism $\dot\varphi : R \to \widehat{S}$ which is formally smoothable, cf. (1.1.2), both the existence result of (1.2) and the uniqueness result of (1.5) can be considerably sharpened.

(1.7) **Proposition.** *Let $R \xrightarrow{\dot\varphi_1} R_1 \xrightarrow{\varphi_1'} \widehat{S}$ and $R \xrightarrow{\dot\varphi_2} R_2 \xrightarrow{\varphi_2'} \widehat{S}$ be Cohen factorizations of a local homomorphism $\dot\varphi : R \to \widehat{S}$.*

If $\dot\varphi_1$ is formally smooth (for example, if the induced field extension $R/\mathfrak{m} \hookrightarrow S/\mathfrak{n}$ is separable), then there exists a comparison υ of $\varphi_1'\dot\varphi_1$ to $\varphi_2'\dot\varphi_2$.

If furthermore both Cohen factorizations are minimal, then any such comparison is an isomorphism.

Proof. Consider the commutative square:

$$\begin{array}{ccc} R & \xrightarrow{\dot\varphi_2} & R_2 \\ {\scriptstyle \dot\varphi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_2'} \\ R_1 & \underset{\varphi_1'}{\longrightarrow} & \widehat{S}\,. \end{array}$$

Assume the left-hand vertical map is a formally smooth homomorphism. As the right-hand vertical map is a surjective homomorphism from a complete local ring, it follows from [9, (19.3.11)] that there exists a homomorphism $\upsilon : R_1 \to R_2$ which makes the diagram commute. Thus, υ is a comparison of $\varphi_1'\dot\varphi_1$ to $\varphi_2'\dot\varphi_2$.

Assume furthermore that both Cohen factorizations are minimal, and consider the induced commutative diagram:

$$\begin{array}{ccc} R_1 & \xrightarrow{\upsilon} & R_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \overline{R}_1 & \xrightarrow{\overline{\upsilon}} & \overline{R}_2 \\ {\scriptstyle \overline{\varphi}'_1}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}'_2} \\ \widehat{S}/\mathfrak{m}\widehat{S} & = & \widehat{S}/\mathfrak{m}\widehat{S} \end{array}$$

where bars denote applications of the functor $k \otimes_R -$.

The minimality of the factorization $\varphi'_2\dot{\varphi}_2$ and the commutativity of the lower square imply an equality $\overline{\mathfrak{m}}_1\overline{R}_1 = \overline{\mathfrak{m}}_2$. As $\overline{\upsilon}$ induces an isomorphism of the residue fields, it follows that $\overline{\upsilon}$ is surjective, cf. [13, (8.4)]. As both rings $\overline{R}_1$ and $\overline{R}_2$ are regular, and have the same embedding dimension by the minimality assumption, the homomorphism $\overline{\upsilon}$ is in fact bijective. It remains to note that on R_1 the $\mathfrak{m}$–adic topology is separated, that on R_2 it is complete, and that R_2 is a flat R–module: under these conditions it follows from [9, (19.7.1.4)] that υ is bijective, as desired. □

The purpose of the following example is to show that the smoothability condition in the preceding proposition cannot be dropped, hence the comparison and uniqueness results of (1.2) and (1.5) are best possible in general.

(1.8) **Example.** Let p be a prime number, let $\mathbb{F}_p$ be the prime field with p elements, and let $k = \mathbb{F}_p(a)$ be a purely transcendental extension of $\mathbb{F}_p$. Furthermore, let $\ell = k(b)$ be an extension of k with $b^{p^3} = a$. For indeterminates X, Y, and Z over ℓ consider the formal power series rings $R_1 = \ell[[X]]$, $R_2 = \ell[[Y]]$, and the homomorphisms of ℓ–algebras $\varphi'_1 : R_1 \to S = \ell[[Z]]/(Z^p)$ and $\varphi'_2 : R_2 \to S = \ell[[Z]]/(Z^p)$ defined by $\varphi'_1(X) = Z + (Z^p) = \varphi'_2(Y)$.

Consider the homomorphism $\dot{\varphi}_1 : R = k \to R_1$, obtained by composing the canonical inclusions $k \to \ell$ and $\ell \to \ell[[X]]$. The equality $\varphi = \varphi'_1\dot{\varphi}_1$ defines a local homomorphism $\varphi : R \to S$, and provides it with a minimal Cohen factorization.

Consider next the homomorphism of $\mathbb{F}_p$–algebras $\mathbb{F}_p[a] \to \ell[[Y]]$ which sends a to $a+Y^{p^2}$. The image of $0 \neq f \in \mathbb{F}_p[a]$ is a polynomial in Y with constant term $f(a) \neq 0$, which is invertible in $\ell[[Y]]$, so that the homomorphism extends to a homomorphism of $\mathbb{F}_p$–algebras $\dot{\varphi}_2 : R \to R_2$. The equalities $\varphi'_2\dot{\varphi}_2(a) = \varphi'_2(a + Y^{p^2}) = a$ show that $\varphi'_2\dot{\varphi}_2$ is a Cohen factorization of φ, and it is clearly minimal.

Claim. *There is no comparison of Cohen factorizations from $\varphi'_1\dot{\varphi}_1$ to $\varphi'_2\dot{\varphi}_2$.*

Indeed, assume that such a comparison $\upsilon : R_1 \to R_2$ exists, and set $c = \upsilon(b)$. One then has:

$$\varphi'_2(c) = \varphi'_2\upsilon(b) = \varphi'_1(b) = \varphi'_2(b)$$

so that Y divides $c - b \in \ell[[Y]]$, and hence Y^{p^3} divides $(c-b)^{p^3} = c^{p^3} - b^{p^3}$. Howevever, the computation:

$$c^{p^3} = \upsilon(b^{p^3}) = \upsilon(a) = \upsilon\dot{\varphi}_1(a) = \dot{\varphi}_2(a) = a + Y^{p^2}$$

shows that $c^{p^3} - b^{p^3} = (a + Y^{p^2}) - a = Y^{p^2}$, and this yields a contradiction. □

We finish this section by spelling out what happens when a Cohen factorization is particularly easy to come by, namely, when φ already has a regular factorization. The results are clear, but they are needed in tracking down the specialization of notions introduced below for arbitrary local homomorphisms to concepts considered earlier for surjective homomorphisms, or more generally for homomorphisms essentially of finite type, cf. (1.0.1).

(1.9) *Remark.* If $R \xrightarrow{\tau} T \xrightarrow{\sigma} S$ is a regular factorization of φ, then $R \xrightarrow{\dot{\tau}} \widehat{T} \xrightarrow{\widehat{\sigma}} \widehat{S}$ is a Cohen factorization of $\dot{\varphi}$, and $\widehat{R} \xrightarrow{\widehat{\tau}} \widehat{T} \xrightarrow{\widehat{\sigma}} \widehat{S}$ is a Cohen factorization of $\widehat{\varphi}$.

The canonical isomorphisms $\widehat{T}/\mathfrak{m}\widehat{T} \cong \widehat{T}/\widehat{\mathfrak{m}}\widehat{T} \cong \widehat{T/\mathfrak{m}T}$ show that $\dot{\tau}$ and $\widehat{\tau}$ have isomorphic closed fibers, which may be identified with the completion of the fiber of τ.

The equality $\operatorname{Ker}\widehat{\sigma} = (\operatorname{Ker}\sigma)\widehat{T}$ identifies the kernel of $\widehat{\sigma}$ with the extension in $\widehat{T}$ of the kernel of σ.

2. Dimension, Depth, and Cohen–Macaulay Defect

The main numerical invariants which measure the "size" of a local ring, R, are its dimension, $\dim R$, and its depth, $\operatorname{depth} R$. Their non-negative difference, $\operatorname{cmd} R = \dim R - \operatorname{depth} R$ is called the *Cohen–Macaulay defect* (or *codepth*) of R, and evaluates the "Cohen–Macaulay character" of R. To any local homomorphism we assign numerical invariants which represent relative versions of these numbers.

(2.1) *Dimension.* This notion is introduced by means of Cohen factorization, due to the following result in which the height of an ideal, $\mathfrak{a}$, is denoted $\operatorname{ht}\mathfrak{a}$.

Lemma. *If $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen factorization of $\dot{\varphi}$, cf. (1.1), then the integer $\dim R' - \dim R - \operatorname{ht}(\operatorname{Ker}\varphi')$ does not depend on the choice of the factorization.*

Proof. By (1.2) it suffices to show that this integer does not change when one replaces the factorization above by $R \xrightarrow{\dot{\varphi}_1} R_1 \xrightarrow{\varphi'_1} \widehat{S}$, where $R_1 = R'/(a_1, \dots, a_r)R'$ and $a_1, \dots, a_r$ is an R'–regular sequence in $\operatorname{Ker}\varphi'$. But then $\dim R_1 = \dim R' - r$, and $\operatorname{ht}(\operatorname{Ker}\varphi'_1) = \operatorname{ht}(\operatorname{Ker}\varphi') - r$. The result follows. □

We set

$$\dim\varphi = \dim R' - \dim R - \operatorname{ht}(\operatorname{Ker}\varphi')$$

and call this number the *dimension* of φ.

Note that, unlike the corresponding notion for rings, the dimension of a homomorphism may take on negative values. Far from being a pathology, this is, in fact, a reflection of the structure of φ:

Example. If φ is surjective, then $\dim \varphi = -\operatorname{ht}(\operatorname{Ker}\varphi)$.

Indeed, the composition $R \to \widehat{R} \xrightarrow{\widehat{\varphi}} \widehat{S}$ in which the first homomorphism is the completion map, provides a Cohen factorization of $\grave{\varphi}$, hence we get:

$$\dim \varphi = \dim \widehat{R} - \dim R - \operatorname{ht}(\operatorname{Ker}\widehat{\varphi}) = -\operatorname{ht}(\operatorname{Ker}\varphi)\ .$$

(2.2) *Depth.* We introduce the *depth* of φ by an *ad hoc* formula:

$$\operatorname{depth}\varphi = \operatorname{depth} S - \operatorname{depth} R\ .$$

This definition may be read as an equality, $\operatorname{depth} S = \operatorname{depth} R + \operatorname{depth}\varphi$, which echoes the naïve expectation that $\operatorname{depth} S$ reflects the genesis of the ring S out of the base R by means of the extension φ. Such a hope is too simplistic to be fulfilled without further conditions on the homomorphism: We shall shortly see that while the desirable inequality $\dim\varphi \geq \operatorname{depth}\varphi$ does hold over Cohen–Macaulay bases, it fails in general. This deficiency might be remedied through some different – and probably more sophisticated – definition of $\operatorname{depth}\varphi$. In case φ has finite flat dimension such a definition is available, and by [6, (5.3)] it agrees with the one adopted here; furthemore, we show in (3.6) below that in this case the "normal" relation holds between dimension and depth.

Example. Let k be a field, set $R = k[[X,Y]]/(X^2, XY)$, and denote by x the image of X in R. If φ is the canonical projection $R \to R/(x) = S$, then by the preceding example we have $\dim\varphi = -\operatorname{ht}(xR) = 0$, while $\operatorname{depth}\varphi = \operatorname{depth} S - \operatorname{depth} R = 1 - 0 = 1$.

This omen notwithstanding, we proceed to introduce a measure of the Cohen–Macaulay character of local homomorphisms by mimicking that for rings.

(2.3) *Cohen–Macaulay Defect.* The difference

$$\operatorname{cmd}\varphi = \dim\varphi - \operatorname{depth}\varphi$$

is called the *Cohen–Macaulay defect* of φ.

Comments on terminology and notation. It might be more in keeping with tradition in Commutative Algebra to consider in place of a local homomorphism $\varphi : R \to S$ the equivalent concept of a *local R–algebra S*. A notation consistent with this point of view would not refer to the structure homomorphism φ, and use $\dim(S|R)$ in place of $\dim\varphi$, etc. We have decided for the present choice of terminology because we feel it represents a slightly different point of view, by giving equal status to the three components of a homomorphism: its target, source, and map. As a bonus we obtain a more flexible notation, which is simpler both to write and to read, compare e.g. the handling of localizations in (3.7) below.

By definition, depth provides an additive function on local homomorphisms. It is easy to see that the dimension or Cohen–Macaulay defect functions do not have this property for arbitrary homomorphisms. Somewhat surprisingly, the next result shows them to be always subadditive.

(2.4) **Theorem.** *For a local homomorphism $\varphi : R \to S$ the following double inequalities hold:*

$$\operatorname{depth} R + \dim \varphi \le \dim S \ \le \dim R + \dim \varphi \ ;$$
$$\operatorname{cmd} \varphi \le \operatorname{cmd} S \le \operatorname{cmd} R + \operatorname{cmd} \varphi \ .$$

Proof. Consider a Cohen factorization of $\dot{\varphi}$ as in (1.1). The double inequality

$$\operatorname{depth} R' \le \dim \widehat{S} + \operatorname{ht}(\operatorname{Ker} \varphi') \le \dim R' \ ,$$

is well known to hold for the surjective homomorphism $\varphi' : R' \to \widehat{S}$.

The right-hand side yields:

$$\begin{aligned} \dim S &= \dim \widehat{S} \\ &\le \dim R' - \operatorname{ht}(\operatorname{Ker} \varphi') \\ &= \dim R + (\dim R' - \dim R - \operatorname{ht}(\operatorname{Ker} \varphi')) \\ &= \dim R + \dim \varphi \ . \end{aligned}$$

As $R'/\mathfrak{m}R'$ is Cohen–Macaulay and $\dot{\varphi}$ is flat, the additivity of dimension and depth on local flat extensions gives:

$$\begin{aligned} \operatorname{depth} R' &= \operatorname{depth} R + \operatorname{depth}(R'/\mathfrak{m}R') \\ &= \operatorname{depth} R + \dim(R'/\mathfrak{m}R') \\ &= \operatorname{depth} R + \dim R' - \dim R \ . \end{aligned}$$

Thus, the left-hand side of the double inequality produces:

$$\begin{aligned} \dim S &= \dim \widehat{S} \\ &\ge \operatorname{depth} R' - \operatorname{ht}(\operatorname{Ker} \varphi') \\ &= \operatorname{depth} R + (\dim R' - \dim R - \operatorname{ht}(\operatorname{Ker} \varphi')) \\ &= \operatorname{depth} R + \dim \varphi \ . \end{aligned}$$

The relations for Cohen–Macaulay defects are consequences of those for dimensions, together with the relevant definitions. □

(2.5) **Corollary.** *If the ring R is Cohen–Macaulay, then there are inequalities:*

$$\dim S = \dim R + \dim \varphi \ ;$$
$$\operatorname{cmd} S = \operatorname{cmd} \varphi \ .$$

□

Some immediate, but important special cases of the preceding corollary deserve separate mention.

(2.6.1) *Remark.* If R is Cohen–Macaulay, then $\operatorname{cmd}\varphi \geq 0$.

(2.6.2) *Remark.* For the local structure homomorphism η_S, cf. (1.0.2), there is an equality $\operatorname{cmd}\eta_S = \operatorname{cmd} S$.

The next result shows that $\dim\varphi$ is a finer invariant than a classical measure of the size of φ, namely, the dimension of its closed fiber. As a consequence, the second inequality of (2.4) is seen to be a sharpening of the standard relation: $\dim S \leq \dim R + \dim(S/\mathfrak{m}S)$, cf. e.g. [13, (15.1.i)].

(2.7) **Theorem.** *For any local homomorphism $\varphi : R \to S$ there is an inequality:*

$$\dim\varphi \leq \dim(S/\mathfrak{m}S)\ .$$

If furthermore $\dim S = \dim R + \dim(S/\mathfrak{m}S)$, then there are equalities:

$$\dim\varphi = \dim(S/\mathfrak{m}S) = \dim S - \dim R\ .$$

Proof. A regular factorization $\acute{\varphi} = \varphi'\dot{\varphi}$ as in (1.1) induces a surjective homomorphism $R'/\mathfrak{m}R' \to \widehat{S}/\mathfrak{m}\widehat{S} = \widehat{S/\mathfrak{m}S}$; denote its kernel by $\bar{\mathfrak{a}}$, and set $\mathfrak{a} = \operatorname{Ker}\varphi'$.
Using the catenarity of the regular ring $R'/\mathfrak{m}R'$ and the flatness of $\dot{\varphi}$, we get:

$$\begin{aligned}\dim(S/\mathfrak{m}S) - \dim\varphi &= \dim(\widehat{S}/\mathfrak{m}\widehat{S}) - \dim\varphi \\ &= (\dim(R'/\mathfrak{m}R') - \operatorname{ht}\bar{\mathfrak{a}}) - (\dim R' - \dim R - \operatorname{ht}\mathfrak{a}) \\ &= \operatorname{ht}\mathfrak{a} - \operatorname{ht}\bar{\mathfrak{a}}\ .\end{aligned}$$

Using the Cohen–Macaulayness of the regular ring $R'/\mathfrak{m}R'$, we can find an $(R'/\mathfrak{m}R')$–regular sequence $b_1 + \mathfrak{m}R', \dots, b_h + \mathfrak{m}R'$ in $\bar{\mathfrak{a}}$, with $h = \operatorname{ht}\bar{\mathfrak{a}}$. As $\bar{\mathfrak{a}} = \mathfrak{a}(R'/\mathfrak{m}R')$, we may furthermore assume the b_i's to be in $\mathfrak{a}$. From the flatness of $\dot{\varphi}$ it follows that the sequence $b_1, \dots, b_h$ is R'–regular, cf. (1.3). Thus, there are inequalities:

$$\operatorname{ht}\mathfrak{a} \geq \operatorname{grade}_{R'} R'/\mathfrak{a} \geq h = \operatorname{ht}\bar{\mathfrak{a}}\ ,$$

and the desired inequality follows.
If $\dim S = \dim R + \dim(S/\mathfrak{m}S)$, then this inequality combines with (2.4) to yield the relations:

$$\dim S - \dim R \leq \dim\varphi \leq \dim(S/\mathfrak{m}S) = \dim S - \dim R\ ,$$

which prove the second assertion of the theorem. □

The concepts introduced in this section acquire a particularly transparent interpretation in the case of flat homomorphisms: This is the contents of the following result, which also establishes that dimension and Cohen–Macaulay defect are additive on flat extensions.

(2.8) **Proposition.** *If φ is a flat local homomorphism, then there are equalities:*

$$\begin{aligned} \dim\varphi = \dim(S/\mathfrak{m}S) &= \dim S - \dim R \ ; \\ \operatorname{depth}\varphi = \operatorname{depth}(S/\mathfrak{m}S) &= \operatorname{depth} S - \operatorname{depth} R \ ; \\ \operatorname{cmd}\varphi = \operatorname{cmd}(S/\mathfrak{m}S) &= \operatorname{cmd} S - \operatorname{cmd} R \ . \end{aligned}$$

Proof. It is a classical fact that for a flat local homomorphism there is an equality $\dim S = \dim R + \dim(S/\mathfrak{m}S)$, cf. e.g. [13, (15.1.ii)]. Thus, the equalities for dimensions result from the preceding proposition.

Comparing the definition of $\operatorname{depth}\varphi$ in (2.2) with the well known equality $\operatorname{depth} S = \operatorname{depth} R + \operatorname{depth}(S/\mathfrak{m}S)$, cf. e.g. [13, (23.3.Corollary)], one gets $\operatorname{depth}\varphi = \operatorname{depth}(S/\mathfrak{m}S)$, and this yields the assertion on depths.

The equalities of Cohen–Macaulay defects are now clear. □

The smooth behavior of the invariants of φ under completions is an easy consequence of the definitions.

(2.9) **Lemma.** *The dimension, depth, and Cohen–Macaulay defect of a local homomorphism φ coincide with the corresponding invariants of its semi-completion, $\grave{\varphi}$, and of its completion, $\widehat{\varphi}$.* □

Finally, we note that the dimension – and hence the Cohen–Macaulay defect – of a factorizable local homomorphism can be computed from any one of its factorizations.

(2.10) **Proposition.** *If $\varphi : R \xrightarrow{\tau} T \xrightarrow{\sigma} S$ is a regular factorization of φ, then*

$$\dim\varphi = \dim T - \dim R - \operatorname{ht}(\operatorname{Ker}\sigma) \ .$$

Proof. As remarked in (1.9), $\grave{\varphi} = \widehat{\sigma}\grave{\tau}$ is a Cohen factorization with $\operatorname{Ker}\widehat{\sigma} = (\operatorname{Ker}\sigma)\widehat{R}$. The expression for the dimension of φ now follows from the equalities $\dim\widehat{T} = \dim T$ and $\operatorname{ht}(\operatorname{Ker}\sigma) = \operatorname{ht}((\operatorname{Ker}\sigma)\widehat{T})$. □

3. Finite Flat Dimension

Blurring somewhat the focus on homomorphisms, we begin this section by establishing an obstruction for finitely generated S–modules to have finite flat dimension over R. Recall that the *flat* (respectively, *projective*) *dimension* of an R–module M is the infimum $\operatorname{fd}_R M$ (respectively, $\operatorname{pd}_R M$) of the lengths of the resolutions of M by flat (respectively, projective) R–modules. The *Cohen–Macaulay defect* of a finitely generated S–module N is defined by $\operatorname{cmd}_S N = \dim_S N - \operatorname{depth}_S N$; thus, $\operatorname{cmd}_S S = \operatorname{cmd} S$.

When R is an algebra over a field, the next result is proved in [5, (2.1)] by using big Cohen–Macaulay modules [12]. The argument given below essentially repeats the one used for the particular case $N = S$ in [3, (5.4)], and relies on (corollaries of) the New Intersection Theorem [15]; the new input is given by (1.1), which shows that *any* local homomorphism is "standard" in the sense of [3].

(3.1) **Theorem.** *If N is a finitely generated S–module, such that $\operatorname{fd}_R N$ is finite, then*

$$\operatorname{cmd} R \le \operatorname{cmd}_S N \ .$$

To get started on the proof we need to know how the crucial assumption of finite flat dimension is affected by passage to a Cohen factorization. This is the contents of the next result.

(3.2) **Lemma.** *Let $\dot\varphi = \varphi'\dot\varphi$ be a Cohen factorization as in* (1.1)*, let N be an S–module, and set $\widehat{N} = N \otimes_S \widehat{S}$. If N is finitely generated, then*

$$\operatorname{fd}_R N = \operatorname{fd}_R \widehat{N} = \operatorname{fd}_{\widehat{R}} \widehat{N} \le \operatorname{pd}_{R'} \widehat{N} = \operatorname{fd}_{R'} \widehat{N} \le \dim R' - \dim R + \operatorname{fd}_R N \ .$$

Proof. The two equalities on the left follow from the faithful flatness of $\widehat{S}$ over S, the natural isomorphisms

$$\operatorname{Tor}_i^R(R/\mathfrak{m}, N) \otimes_S \widehat{S} \cong \operatorname{Tor}_i^R(R/\mathfrak{m}, \widehat{N}) \cong \operatorname{Tor}_i^{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}}, \widehat{N}) \ ,$$

and the extension of the classical "local criterion for flatness" by the formula

$$\operatorname{fd}_R N = \sup\{i \mid \operatorname{Tor}_i^R(R/\mathfrak{m}, N) \neq 0\} \ ,$$

which may be found in [1, (2.57)].

The first inequality is immediate from the fact that R' is R–flat.

The equality $\operatorname{pd}_{R'} \widehat{N} = \operatorname{fd}_{R'} \widehat{N}$ is due to the finite generation of $\widehat{N}$ over R'.

The second inequality results from [5, (2.7)], applied to the regular factorization $\dot\varphi = \varphi'\dot\varphi$. □

Proof of (3.1). Let $\dot\varphi = \varphi'\dot\varphi$ be a Cohen factorization. It is well known that $\operatorname{cmd}_S N = \operatorname{cmd}_{\widehat{S}} \widehat{N} = \operatorname{cmd}_{R'} \widehat{N}$. As $\dot\varphi$ is flat with Cohen–Macaulay fiber, we also have $\operatorname{cmd} R' = \operatorname{cmd} R$. Furthermore, by the preceding lemma the projective dimension of $\widehat{N}$ over R' is finite. There is then an inequality $\dim R' \le \dim_{R'} \widehat{N} + \operatorname{pd}_{R'} \widehat{N}$: as shown in [14, pp. 69 and 74] this is a consequence of the New Intersection Theorem, proved in [15]. Rewriting the last inequality by means of the Auslander–Buchsbaum Equality, $\operatorname{pd}_{R'} \widehat{N} = \operatorname{depth} R' - \operatorname{depth}_{R'} \widehat{N}$, we obtain the desired expression. □

(3.3) *Flat Dimension of a Homomorphism.* We set $\operatorname{fd} \varphi = \operatorname{fd}_R S$, and call this number the *flat dimension* of φ. The (in-)equalities of (3.2) yield

$$\operatorname{fd} \varphi = \operatorname{fd} \dot\varphi = \operatorname{fd} \widehat{\varphi} \le \operatorname{fd} \varphi' \le \dim R' - \dim R + \operatorname{fd} \varphi \ .$$

It follows, in particular, that $\operatorname{fd} \varphi'$ is finite if and only if $\operatorname{fd} \varphi$ is finite. When this is the case, new relations hold between the invariants of φ introduced in the preceding section.

First we complete the inequalities of (2.4).

(3.4) **Theorem.** *If $\varphi : R \to S$ is a local homomorphism of finite flat dimension, then the following inequalities hold:*

$$\dim R + \operatorname{depth}\varphi \le \dim S \ ;$$
$$\operatorname{cmd} R \le \operatorname{cmd} S \ .$$

Proof. The second inequality is a restatement of (3.1) when $N = S$, and is equivalent to the first one by the definition of $\operatorname{depth}\varphi$ in (2.2). □

Next we complement the expression for the dimension of φ obtained in (2.10). For an arbitrary ideal $\mathfrak{a}$ in a local ring T, the difference $\operatorname{pd}_T(T/\mathfrak{a}) - \operatorname{grade}_T(T/\mathfrak{a})$ is non-negative by a classical result of Rees, cf. [13, p. 132]. We call it the *imperfection* of $\mathfrak{a}$, and denote it by $\operatorname{imp}\mathfrak{a}$.

(3.5) **Proposition.** *If $R \overset{\tau}{\to} T \overset{\sigma}{\to} S$ is a regular factorization of φ, and $\operatorname{fd}\varphi$ is finite, then $\operatorname{fd}\sigma$ is finite, and*

$$\dim\varphi = \dim T - \dim R - \operatorname{grade}_T S \ ;$$
$$\operatorname{depth}\varphi = \dim T - \dim R - \operatorname{pd}_T S \ ;$$
$$\operatorname{cmd}\varphi = \operatorname{imp}(\operatorname{Ker}\sigma) \ .$$

In particular, if φ is surjective, then $\dim\varphi = -\operatorname{grade}_R S$, $\operatorname{depth}\varphi = -\operatorname{pd}_R S$, and $\operatorname{cmd}\varphi = \operatorname{imp}(\operatorname{Ker}\varphi)$.

Proof. The finiteness of $\operatorname{fd}\sigma$ is seen from (1.9) and (3.3).

The expression for $\dim\varphi$ then follows by substituting in (2.10) the equality $\operatorname{ht}(\operatorname{Ker}\sigma) = \operatorname{grade}_T S$, cf. [4]. The expression for $\operatorname{depth}\varphi$ is a consequence of its definition in (2.2) and the Auslander–Buchsbaum Equality.

In the surjective case, consider the regular factorization $\varphi = \varphi \circ \operatorname{id}_R$. □

As $\operatorname{cmd}\varphi = \operatorname{cmd}\grave{\varphi}$ by (2.9), the proposition applied to a regular factorization of $\grave{\varphi}$ yields:

(3.6) **Corollary.** *If $\operatorname{fd}\varphi < \infty$, and $\grave{\varphi} = \varphi'\dot{\varphi}$ is a Cohen factorization, then*

$$\operatorname{cmd}\varphi = \operatorname{imp}(\operatorname{Ker}\varphi') \ .$$

It follows in particular that $\operatorname{imp}(\operatorname{Ker}\varphi')$ does not depend on the choice of the factorization, and that $\operatorname{cmd}\varphi$ is non-negative. □

Another application of the proposition is to localization of Cohen–Macaulay defects. Let $\mathfrak{q}$ be a prime ideal in the local ring S. We want to compare the Cohen–Macaulay defect of φ with that of the induced local homomorphism

$$\varphi_{\mathfrak{q}} : R_{\mathfrak{q}\cap R} \longrightarrow S_{\mathfrak{q}} \ ,$$

where $\mathfrak{q} \cap R$ denotes the inverse image of $\mathfrak{q}$ in R; we call $\varphi_{\mathfrak{q}}$ the *localization* of φ at $\mathfrak{q}$.

Unlike the case of rings, where there always is an inequality $\operatorname{cmd} S_{\mathfrak{q}} \le \operatorname{cmd} S$, the corresponding inequality $\operatorname{cmd}\varphi_{\mathfrak{q}} \le \operatorname{cmd}\varphi$ may fail even for flat homomorphisms. A detailed investigation of the localization problem for Cohen–Macaulay defects of homomorphisms of finite flat dimension is carried out in [4]. Here we treat an important special case.

(3.7) **Proposition.** *Let $\varphi : R \to S$ be a local homomorphism, and let $\mathfrak{q}$ be a prime ideal of S. If φ is essentially of finite type and of finite flat dimension, then $\varphi_{\mathfrak{q}}$ has the same properties, and there is an inequality:*

$$\operatorname{cmd} \varphi_{\mathfrak{q}} \leq \operatorname{cmd} \varphi \ .$$

Proof. It is clear from the definitions that $\varphi_{\mathfrak{q}}$ is essentially of finite type, and it is a standard consequence of the flatness of localizations that its flat dimension is finite.

Consider a factorization $\varphi = \sigma\tau$ as in (1.0.1) and set $\mathfrak{p}' = \mathfrak{q} \cap T$, $\mathfrak{p} = \mathfrak{p}' \cap R$. The decomposition $\varphi_{\mathfrak{q}} = \sigma_{\mathfrak{q}}\tau_{\mathfrak{p}'}$ then yields a regular factorization of $\varphi_{\mathfrak{q}}$. By (3.5) the proposition would follow from an inequality $\operatorname{imp}(\operatorname{Ker} \sigma_{\mathfrak{q}}) \leq \operatorname{imp}(\operatorname{Ker} \sigma)$. As σ is surjective, we have $\operatorname{Ker} \sigma_{\mathfrak{q}} = (\operatorname{Ker} \sigma)_{\mathfrak{q}}$. The inequality $\operatorname{imp}(\operatorname{Ker} \sigma)_{\mathfrak{q}} \leq \operatorname{imp}(\operatorname{Ker} \sigma)$ is clear, since $\operatorname{pd}_{T_{\mathfrak{p}'}} S_{\mathfrak{q}} \leq \operatorname{pd}_T S$, and $\operatorname{grade}_{T_{\mathfrak{p}'}} S_{\mathfrak{q}} \geq \operatorname{grade}_T S$. □

We now single out those local homomorphisms which provide a relative version of the local Cohen–Macaulay rings.

(3.8) *Cohen–Macaulay Local Homomorphisms.* Recall that an ideal $\mathfrak{a}$ of a local ring T is called *perfect* if $\operatorname{pd}_T(T/\mathfrak{a}) = \operatorname{grade}_T(T/\mathfrak{a})$ (that is, if $\operatorname{imp} \mathfrak{a} = 0$).

(3.8.1) **Proposition.** *The following conditions on a local homomorphism, $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$, are equivalent:*

(*i*) $\operatorname{fd} \varphi$ *is finite and* $\operatorname{cmd} \varphi = 0$.
(*ii*) $\dot{\varphi}$ *has a Cohen factorization* $\dot{\varphi} = \varphi'\dot{\varphi}$ *with* $\operatorname{Ker} \varphi'$ *a perfect ideal.*
(*iii*) *In each Cohen factorization* $\dot{\varphi} = \varphi'\dot{\varphi}$ *the ideal* $\operatorname{Ker} \varphi'$ *is perfect.*

Proof. It is clear that (*i*) implies (*iii*) by (3.6) and that (*iii*) implies (*ii*) by (1.1). To see that (*ii*) implies (*i*), note first that if $\operatorname{Ker} \varphi'$ is perfect, then $\operatorname{fd} \varphi' = \operatorname{pd}_{R'} \widehat{S}$ is finite; thus, (3.6) applies, and shows that $\operatorname{cmd} \varphi = \operatorname{imp}(\operatorname{Ker} \varphi') = 0$. □

A local homomorphism φ which satisfies the conditions of the proposition is said to be *Cohen–Macaulay at* $\mathfrak{n}$. We note some familiar forms of the new concept.

(3.8.2) *Remark.* The local structure homomorphism η_S of a local ring $(S, \mathfrak{n})$ is Cohen–Macaulay at $\mathfrak{n}$ if and only if the ring S is Cohen–Macaulay: this follows from (2.6.2). Thus, the notion of Cohen–Macaulay homomorphism contains that of Cohen–Macaulay ring as "the absolute case".

(3.8.3) *Remark.* A flat local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is Cohen–Macaulay at $\mathfrak{n}$ if and only if the closed fiber $S/\mathfrak{m}S$ is a Cohen–Macaulay ring, cf. (2.8).

(3.8.4) *Remark.* If $\varphi = \sigma\tau$ is a regular factorization, then φ is Cohen–Macaulay at $\mathfrak{n}$ if and only if the ideal $\operatorname{Ker} \sigma$ is perfect: this is a consequence of (3.5). In particular, a surjective homomorphism is Cohen–Macaulay precisely when its kernel is a perfect ideal.

(3.8.5) *Remark.* If φ is essentialy of finite type and Cohen–Macaulay at $\mathfrak{n}$, then for any prime ideal $\mathfrak{q}$ of S the induced homomorphism $\varphi_{\mathfrak{q}}$ is Cohen–Macaulay at $\mathfrak{q}S_{\mathfrak{q}}$: this follows from (3.7). However, the situation is much more complicated in general: For instance the completion map $R \to \widehat{R}$ is obviously Cohen–Macaulay at $\mathfrak{m}\widehat{R}$ by (3.8.3), but its localizations do not always inherit this property, cf. [4] for details.

The next statement is a parallel to (2.5), with the Cohen–Macaulay hypothesis switched from the base ring to the homomorphism; the similarity of the conclusions obtained is quite remarkable.

(3.9) **Proposition.** *If a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is Cohen–Macaulay at $\mathfrak{n}$, then there are equalities:*

$$\begin{aligned} \dim S &= \dim R + \dim \varphi \ ; \\ \operatorname{cmd} S &= \operatorname{cmd} R \ . \end{aligned}$$

Proof. It suffices to combine the corresponding inequalities of (2.4) and (3.4). □

Now we can complete the solution of the problem of descending *and* ascending Cohen–Macaulayness of local rings by homomorphisms of finite flat dimension, cf. [5, Section 2]. The result is a perfect generalization of the well known fact that a flat extension S of R is Cohen–Macaulay if and only if both R and $S/\mathfrak{m}S$ are, cf. e.g. [13, (23.3.Corollary)]; indeed, by (3.8.3) the Cohen–Macaulay properties of φ and of $S/\mathfrak{m}S$ are equivalent when φ is flat.

(3.10) **Theorem.** *If $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism, then the following are equivalent:*

(*i*) *R is Cohen–Macaulay and φ is Cohen–Macaulay at $\mathfrak{n}$.*
(*ii*) *S is Cohen–Macaulay and φ has finite flat dimension.*

Proof. If R and φ are Cohen–Macaulay, then so is S by the second equality in the preceding proposition, and $\operatorname{fd} \varphi$ is finite by definition.

Conversely, let $\operatorname{cmd} S = 0$; then $\operatorname{cmd} \varphi = 0$ by (2.4); as $\operatorname{fd} \varphi < \infty$ is also assumed in (*ii*), it follows that φ is Cohen–Macaulay at $\mathfrak{n}$, and the equality $\operatorname{cmd} R = 0$ results from another application of (3.9). □

A particular type of Cohen–Macaulay homomorphisms has been identified earlier, in an entirely different context.

(3.11) *Gorenstein Local Homomorphisms.* A local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is called *Gorenstein at* $\mathfrak{n}$, if $\operatorname{fd} \varphi$ is finite and there is an integer d such that an equality of Bass numbers $\mu_R^i = \mu_S^{i+d}$ holds for $i \in \mathbb{Z}$ (in [2] and [3] such a φ is called a *Gorenstein local homomorphism*). These homomorphisms are shown in [*loc. cit.*] to provide a relative version of the class of Gorenstein local rings. In order to characterize them in terms of Cohen factorization, we recall that an ideal in a local ring is called *Gorenstein* if it is perfect, and the last non-zero module in its minimal free resolution is cyclic.

Proposition. *The following conditions on a local homomorphism, $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$, are equivalent:*

(*i*) *φ is Gorenstein at $\mathfrak{n}$.*
(*ii*) *$\dot{\varphi}$ has a Cohen factorization $\dot{\varphi} = \varphi'\dot{\varphi}$ with $\operatorname{Ker}\varphi'$ a Gorenstein ideal.*
(*iii*) *In each Cohen factorization $\dot{\varphi} = \varphi'\dot{\varphi}$ the ideal $\operatorname{Ker}\varphi'$ is Gorenstein.*

Proof. (*i*) $\Rightarrow$ (*iii*). If φ is Gorenstein, then $\operatorname{fd}\varphi < \infty$ by definition, hence $\operatorname{fd}\varphi'$ is finite by (3.3). On the other hand, $\operatorname{fd}\dot{\varphi} = 0$ by assumption. Thus, φ' is Gorenstein by the decomposition property for Gorenstein local homomorphisms, cf. [3, (4.6.b)], and we conclude that the ideal $\operatorname{Ker}\varphi'$ is Gorenstein by applying [*idem*, (4.3)].

(*iii*) $\Rightarrow$ (*ii*) : Follows from (1.1).

(*ii*) $\Rightarrow$ (*i*). The local homomorphism $\dot{\varphi}$ is flat with Gorenstein closed fiber, hence is Gorenstein by [3, (4.2)]. The homomorphism φ' is surjective with kernel a Gorenstein ideal, hence is Gorenstein by [*idem*, (4.3)]. By the composition property of Gorenstein local homomorphisms, cf. [*idem*, (4.6.a)], we conclude that $\dot{\varphi} = \varphi'\dot{\varphi}$ is Gorenstein, and clearly this implies that φ has the same property. □

The first assertion of the next result now follows by simple comparison of the statements of (3.11) and (3.8.1); the second assertion then comes from (3.9), and generalizes [3, (5.5)].

(3.12) **Corollary.** *A local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ which is Gorenstein at $\mathfrak{n}$ is also Cohen–Macaulay at $\mathfrak{n}$.*

In particular, for such a φ there is an equality $\operatorname{cmd} R = \operatorname{cmd} S$. □

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