LOCAL RINGS OVER WHICH ALL MODULES HAVE RATIONAL POINCARÉ SERIES

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To Steve Halperin on February 1

Abstract. If the homotopy Lie algebra $\pi^*(R)$ of a local ring R contains a free Lie subalgebra of finite codimension, then for each finitely generated R–modules M the Poncaré series $P^R_M(t) = \sum_{n=0}^{\infty} \dim_k \operatorname{Tor}^R_n(M,k) \cdot t^n$ represents a rational function in t, and there is a least common denominator for all these functions. When this denominator is a power of $(1-t)$, the ring R is a complete intersection, which has at most one non-quadratic defining equation.

Introduction

Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}_R$ and residue field k, and let M be a finitely generated R–module. An important characteristic of M is contained in its *Betti sequence* $\{ b^R_n(M) = \dim_k \operatorname{Tor}^R_n(M,k) \}_{n \geq 0}$, which refines the numerical information given by its minimal number of generators, $b^R_0(M)$. Betti sequences are known to be primitive recursive [27], but even when $M = k$ they need not satisfy a recurrence relation with constant coefficients [1].

Empirical evidence suggests that the nature of the singularity of R is reflected in the asymptotic patterns of the Betti sequences of its modules. In particular, the reference provided with each of the following conditions:

(a) R is a complete intersection, [21];
(b) R is one link from a complete intersection, [12];
(c) R is two links from a complete intersection and R is Gorenstein, [12];
(d) $\operatorname{edim} R - \operatorname{depth} R \leq 3$, [12];
(e) $\operatorname{edim} R - \operatorname{depth} R = 4$ and R is Gorenstein, [26];

establishes that if R has the corresponding property then there exist integers $a_1, \dots, a_d$ such that a recurrent relation:

$$b^R_n(M) = a_1 b^R_{n-1}(M) + \cdots + a_d b^R_{n-d}(M)$$

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holds for any R–modules M and all $n \gg 0$. In other words this means that for each M the formal power series:

$$P_M^R(t) = \sum_{n=0}^{\infty} b_n^R(M) \cdot t^n \,,$$

known as its *Poincaré series*, represents a rational function with denominator $\operatorname{Den}^R(t) = 1 - a_1 t - \cdots - a_d t^d$.

In case (a) the result follows from Gulliksen's [21] discovery that $\operatorname{Tor}_*^R(M\,,k)$ then has a structure of artinian graded module over a certain graded polynomial ring. In the remaining cases R is shown to be a homomorphic image of a complete intersection by a Golod ideal, and rationality is deduced from a result of Levin implicit in [31], cf. [12], whose proof uses that structure along with techniques of Golod homomorphisms.

In Section 1 we generalize the results of Gulliksen and Levin and prove that rationality can be deduced from information encoded in an intrinsic invariant of the ring R – its *homotopy Lie algebra* $\pi^*(R)$, cf. [5]. The vector space dimension $e_i = \dim_k \pi^i(R)$ is called the *i'th deviation* of R and may be computed from the equality:

$$P_k^R(t) = \prod_{i=0}^{\infty} \frac{(1+t^{2i+1})^{e_{2i+1}}}{(1-t^{2i+2})^{e_{2i+2}}} \,.$$

As a consequence of (1.5) and (1.7) below we have:

Theorem A. *If $\pi^*(R)$ contains a free graded Lie subalgebra of finite codimension, then there exists a polynomial $\operatorname{Den}^R(t) \in \mathbb{Z}[t]$ such that for any finitely generated R–module M the series $\operatorname{Den}^R(t) \cdot P_M^R(t)$ is a polynomial with integer coefficients.*

Furthermore, if the free subalgebra contains $\pi^j(R)$ for $j > n$, then $\operatorname{Den}^R(t) \cdot P_k^R(t)$ divides $\prod_{0 \leqslant 2i+1 \leqslant n} (1+t^{2i+1})^{e_{2i+1}}$. □

By the dictionary of [5] and [11], homotopy Lie algebras of local rings correspond to rational homotopy Lie algebras of one-connected finite CW complexes: The translation of Theorem A into one of homotopy theory is carried out in [9]. A version of the theorem, proved in (1.6), has been used to add to the list above a new class of rings with rational Poincaré series for all modules:

(f) $\operatorname{edim} R - \operatorname{depth} R = 4$ and R is a Cohen-Macaulay almost complete intersection which contains $\frac{1}{2}$, [29].

It is also shown in [29] that this class contains rings which are not Golod images of complete intersections, hence over which rationality does not follow from earlier results.

We list some applications of rationality.

• Let R be essentially of finite type over a field ℓ of characteristic zero. If the cotangent homology groups $T_i(R|\ell\,, R)$ vanish for $i \gg 0$, then Herzog [25] has proved that the rationality of Poincaré series of all R–modules implies that R is a complete intersection; his conjecture that this holds in general is still open.

• Eisenbud [17] has proved that over a complete intersection the minimal resolution of a modules with bounded Betti numbers is eventually periodic of period 2, and he has conjectured that this holds over all rings; although the conjecture fails in general [18], precise information on $\operatorname{Den}^R(t)$ has been used for its proofs in cases (b) through (e) in [8] and (f) in [29].

• The form of the expressions for $\operatorname{Den}^R(t)$ for rings R satisfying one of the conditions (b) through (f) has been recently used by Sun [40] to show that all finitely generated modules over such R have eventually non-decreasing Betti sequences; while this property also holds in case (a) by [10], it is an open problem whether it is valid over arbitrary rings, cf. [5].

• If $P^R_M(t)$ is rational, then the growth of the sequence $\{\beta^R_n(M) = \sum_{i=0}^n b^R_n(M)\}_{n\geq 0}$ is either polynomial or exponential [8]; this dichotomy also holds [5] for $M = k$ over arbitrary R, but it is not known whether other asymptotic patterns occur in general, cf. [8].

In Section 2 we study the asymptotic behavior of the *Betti functions* $b^R_M : n \mapsto b^R_n(M)$ as a measure of the complexity of the ring. The starting point is Gulliksen's result [23] that if b^R_k is bounded by some polynomial in n, then R is a complete intersection.

For such rings, it has been known since Tate [41] that b^R_k is indeed a polynomial of degree $c-1$, where c denotes the codimension of R, that is, the least integer such that the $\mathfrak{m}_R$–adic completion $\widehat{R}$ can be written in the form $Q/(f_1,\dots,f_c)$ with a regular local ring Q and a Q–regular sequence $f_1,\dots,f_c$ contained in $\mathfrak{m}_Q^2$. Gulliksen [21] has proved that the Poincaré series of any M over a codimension c complete intersection R s rational with denominator $(1-t^2)^c$. Equivalently, the subsequences of even Betti numbers and of odd Betti numbers are each eventually given by a polynomial of degree at most $c-1$. Furthermore, it is shown in [7] that the degree and leading coefficient of these polynomials are equal. Note that when $c=0$ these results say that R is regular if and only if any M has finite projective dimension, which is the Auslander–Buchsbaum–Serre characterization of regularity. When $c=1$ they coincide with Eisenbud's characterization of hypersurface rings as those R over which all Betti functions become eventually constant.

The information available so far raises the possibility that over complete intersections all Betti functions are eventually polynomial in n. As it turns out, this is rarely the case, and imposes conditions on the defining relations of $\widehat{R}$.

In order to formulate our result succintly, we note that for any local ring R of embedding dimension $e = \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$, the integer $q(R) = \binom{e+1}{2} - \dim_k \mathfrak{m}_R^2/\mathfrak{m}_R^3$ is non-negative. It measures "the number of quadratic relations" of R, and the quotation marks can be dropped if R is (the localization at the irrelevant ideal of) a graded k–algebra.

When R is a complete intersection of codimension c we have $q(R) \leq c$. In the graded case the equality $q(R) = c$ may be expressed either by the condition that the f_i's can all be chosen to be quadratic forms in some minimal set of generators of $\mathfrak{m}_Q$, or by the requirement that R has multiplicity 2^c. Although these conditions are no longer equivalent in general (the complete intersection R defined in $k[[X,Y]]$ by $f_1 = X^2$ and $f_2 = XY + Y^3$ has $q(R) = 2$ and multiplicity 5), the multiplicity of any codimension c complete intersection is at least 2^c, cf. [14, §7, Proposition 7],

In view of the preceding discussion, the next result is a consequence of (2.1) and (2.3).

Theorem B. *If R is a local ring such that for every finitely generated R–module M the function $b^R_M : n \mapsto b^R_n(M)$ is eventually given by some polynomial in n, then R is a complete intersection of codimension c, with $c \leq q(R)+1$.*

If R is a complete intersection of minimal multiplicity, then the Betti function of any finitely generated R–module is eventually polynomial. □

The graded version of the theorem acquires a particularly simple form:

Theorem B´. *A necessary condition for all finitely generated graded modules over a graded ring R to have eventually polynomial Betti functions is that R be a complete intersection with at most one non-quadratic relation, and a sufficient condition is that it be a complete intersection of quadrics.* □

Except for the codimension one situation, I do not know whether the remaining case of a complete intersection with *exactly* one non-quadratic relation belongs to either category.

1. Generalized Golod rings

In (1.1) through (1.3) we introduce notation and recall some facts used in the sequel. In (1.4) and (1.5) we establish rationality results in terms of distinguished Golod algebras, by extending techniques of Gulliksen and Levin. For applications to the rings described in (b) through (f) of the Introduction, it has been necessary to draw rationality conclusions from hypotheses on the algebra structure of Koszul homology; the corresponding adjustments are carried out in (1.6). In (1.7) we recast the rationality results in terms of the homotopy Lie algebra of R, and in (1.8) we relate them to earlier work.

(1.1) Acyclic closures. In this paper, the name *DG R–algebras* will be reserved for differential graded algebras U which are non-negatively graded, associative, commutative (in the strict sense, that is, squares of element of odd degree are zero), whose degree zero component is a homomorphic image of R, and whose homology $\mathrm{H}(U)$ has $\mathrm{H}_0(U) \neq 0$ and $\mathrm{H}_n(U)$ a finitely generated R–module for each n. We denote by $(\ )^\natural$ the functor forgetful of differentials, and by $|\ |$ the degree of a homogeneous element.

Tate [41] has described a process of "killing cycles by adjunction of divided powers variables"; henceforth we refer to such variables by the name *Γ–variables*, and write $\gamma^i(y)$ for the i'th divided power of a Γ–variable x. The result of adjoining a set X of Γ–variables to a DG algebra U is denoted $U\langle X\rangle$. We set $X_n = \{\, x \in X \mid |x| = n \,\}$ and $X_{\leq n} = \{\, x \in X \mid |x| \leq n \,\}$.

A DG algebra of the form $U\langle X\rangle$ where the set X_n is empty when n is not positive, $\partial(X_1)$ is a minimal set of generators of $\mathfrak{m}_R U_0$ modulo $\partial(U_1)$, and $\partial(X_n)$ is a basis of the k–vector space $\mathrm{H}_{n-1}(U\langle X_{<n}\rangle)$ when $n \geq 2$, is called an *acyclic closure* of U. It is clear that acyclic closures always exist. It is not difficult to see that for them the set X_n is finite for each degree n, and that they are unique up to (non-canonical) isomorphisms of DG R–algebras: for details cf. [24]. Gulliksen [24, (1.6.2)] has proved that an acyclic closure $U\langle X\rangle$ satisfies $\partial(U\langle X\rangle) \subseteq \mathfrak{m}_U(U\langle X\rangle)$, where $\mathfrak{m}_U$ is the kernel of the extension $\varepsilon_U : U \to k$ of the canonical homomorphism $\varepsilon_R : R \to k$.

In particular, when R is viewed as a DG R–algebra in the obvious way, its acyclic closure is a minimal free resolution of the R–module k.

(1.2) Golod algebras. Let U be a DG R–algebra with $H_0(U) = k$, and let $B = \{ h_i \}_{i\geq 1}$ be a (homogeneous) basis of the k–vector space $\operatorname{Ker}\big(\varepsilon_{\mathrm{H}(U)} : \mathrm{H}(U) \to k\big)$. A function μ from the set of finite sequences of elements of B to $\operatorname{Ker}\varepsilon_U$ is said to be a *trivial Massey operation*, if:

$$\mu(h_i) = z_i \text{ with } \partial(z_i) = 0 \text{ and } \operatorname{cls}(z_i) = h_i\,;$$
$$\partial\mu(h_{i_1},\dots,h_{i_n}) = \sum_{j=1}^{n} \overline{\mu(h_{i_1},\dots,h_{i_j})}\mu(h_{i_{j+1}},\dots,h_{i_n})\,,$$

where $\overline{a}$ stands for $(-1)^{|a|+1}a$.

A DG algebra U as above, which has a basis B admitting a trivial Massey operation, is called a *Golod algebra*.

If U is such an algebra, then let V be a graded free R–module with basis $\{ v_i \mid |v_i| = |h_i| + 1 \}$, and let $T(V)$ denote the tensor algebra of V over R. An R–linear, degree -1 endomorphism ∂ of the graded R–module $U \otimes_R T(V)$ is defined by the formula:

$$\partial(u\otimes v_{i_1}\otimes\cdots\otimes v_{i_n}) = \partial(u)\otimes v_{i_1}\otimes\cdots\otimes v_{i_n} + (-1)^{|u|}u\otimes\sum_{j=1}^{n}\mu(h_{i_1},\dots,h_{i_j})\otimes v_{i_{j+1}}\otimes\cdots\otimes v_{i_n}\,.$$

This construction, which goes back to Golod [20], has the property that $\partial^2 = 0$, so that $U \otimes_R T(V)$ becomes a left DG U–module, whose homology is equal to k.

(1.3) Distinguished DG algebras. Let U be a DG R–algebra such that:

(1.3.1) $\mathrm{H}_0(U) = k$.

(1.3.2) $U = R\langle Y\rangle$ for a finite set of Γ–variables $Y = Y_{\geq 1}$, linearly ordered so that $y' < y$ implies $|y'| \leq |y|$.

For any finitely generated R–module M a *Hilbert series* $|\mathrm{H}(M \otimes_R U)|(t)$ is defined: here and below, for a graded vector space W such that $\dim_k W_n$ is finite for all n and zero for $n \ll 0$, we denote by $|W|(t)$ the formal Laurent series $\sum_{n\in\mathbb{Z}} \dim_k W_n t^n$. Gulliksen has introduced a powerful technique for the study of $\mathrm{H}(M \otimes_R U)$, based on the action of homogeneous R–linear endomorphisms $\theta : U \to U$ such that:

$$\theta(uv) = (-1)^{|\theta||v|}\theta(u)v + u\theta(v)\,;$$
$$\theta(\gamma^i(y)) = \gamma^{i-1}(y)\theta(y) \text{ when } |y| \text{ is even}\,;$$
$$\theta\partial(u) = (-1)^{|\theta|}\partial\theta(u)\,.$$

Such θ will be called *DGΓ–derivations* of U. If $\{ \theta_s \}_{s\in S}$ is a family of DGΓ–derivations, then by setting $\Theta_s u = \theta_s(u)$ for $s \in S$, one obtains on $M \otimes_R U$ a structure of left DG module over the tensor algebra of a free graded R–module with basis $\{ \Theta_s \}_{s\in S}$.

Consider now a DG algebra U as above, which also satisfies:

(1.3.3) There are DGΓ–derivations $\{ \theta_y \}_{y\in Y}$ of U such that $\theta_y(y) = 1$ and $\theta_y(y') = 0$ for $y' \in Y$ with $y' < y$.

The proof of [22, Theorem 2], establishes that:

If U satisfies conditions (1.3.1) *through* (1.3.3) *and* $\mathcal{T}$ *denotes the tensor algebra* $\mathcal{T}$ *of a free graded R–module with basis* $\{\,\Theta_y\,\}_{y\in Y\,,|y|\text{ even}}$, *then for any graded* $\mathcal{T}$*–submodule* C *of* $\mathrm{H}(M\otimes_R U)$ *the series* $\prod_{|y|\text{ even}}(1-t^{|y|})\cdot|C|(t)$ *is a polynomial with integer coefficients.*

Using the terminology of [31] in a slightly restricted meaning, we say a DG R–algebra is *distinguished*, if it satisfies the three conditions above, along with:

(1.3.4) An acyclic closure $U\langle X\rangle$ of U has $\partial(U\langle X\rangle)\subseteq\mathfrak{m}_R U\langle X\rangle$.

The next result is folklore.

(1.3.5) **Lemma.** *The DGΓ–derivations* $\{\,\theta_y\,\}_{y\in Y}$ *of a distinguished DG R–algebra* $U=R\langle Y\rangle$ *can be extended to DGΓ–derivations of its integral closure* $U\langle X\rangle=R\langle Y\rangle\langle X\rangle$.

Proof. Starting from the trivial case $U\langle X_{\le 1}\rangle=U$, we can assume by induction that for some $n\ge 1$ the derivations $\{\,\theta_y\,\}_{y\in Y}$ have been extended to DGΓ–derivations of $U\langle X_{\le n}\rangle$, denoted by the same letters. We want to extend them further to DGΓ–derivations of $U\langle X_{\le n+1}\rangle$. It is easy to see, cf. e.g. [24, (1.3.4)], that a necessary and sufficient condition for this is that for $y\in Y$ there be inclusions:

$$\theta_y(\mathrm{Z}_n(U\langle X_{\le n}\rangle))\subseteq\partial(U\langle X_{\le n}\rangle_{n+1-|y|})\,,$$

where $\mathrm{Z}(\ )$ denotes the graded R–submodule of cycles.

If $1\le|y|\le n-1$, then $\mathrm{H}_{n-|y|}(U\langle X_{\le n}\rangle))=0$, hence:

$$\mathrm{Z}_{n-|y|}(U\langle X_{\le n}\rangle)=\partial((U\langle X_{\le n}\rangle)_{n+1-|y|})\,.$$

As θ_y commutes with the differential of $U\langle X_{\le n}\rangle$, we also have:

$$\theta_y(\mathrm{Z}_n(U\langle X_{\le n}\rangle))\subseteq\mathrm{Z}_{n-|y|}(U\langle X_{\le n}\rangle)\,.$$

If $|y|=n$, then from (1.3.4) we get:

$$\mathrm{Z}_n(U\langle X_{\le n}\rangle)=(U\langle X_{\le n}\rangle)_n\cap\partial(U\langle X\rangle)\subseteq(U\langle X_{\le n}\rangle)_n\cap(\mathfrak{m}_R U\langle X\rangle)=\mathfrak{m}_R(U\langle X_{\le n}\rangle)_n\,,$$

and this implies:

$$\theta_y(\mathrm{Z}_n(U\langle X_{\le n}\rangle))\subseteq\theta_y(\mathfrak{m}_R(U\langle X_{\le n}\rangle)_n)\subseteq\mathfrak{m}_R U_0=\partial(U_1)\,.$$

Finally, if $|y|\ge n+1$, then $\theta_y(\mathrm{Z}_n(U\langle X_{\le n}\rangle))\subseteq U_{n-|y|}=0$. □

(1.4) Rational Poincaré series. The proof of the following rationality result uses techniques of [22], [19], [31], and avoids the problems arising in the latter paper from considerations of Yoneda product structures.

Theorem. *Let R be a local ring with a distinguished Golod algebra $U = R\langle y_1, \dots, y_m\rangle$.*
There is then a polynomial $\operatorname{Den}^R(t) \in \mathbb{Z}[t]$ such that the Poincaré series of every finitely generated R–module M can be written in the form:

$$P_M^R(t) = \frac{p_M(t)}{\operatorname{Den}^R(t)}$$

with $p_M(t) \in \mathbb{Z}[t]$. Furthermore:

$$P_k^R(t) = \frac{\prod_{0 \leqslant 2i+1 \leqslant n} (1 + t^{2i+1})^{d_{2i+1}}}{\operatorname{Den}^R(t)},$$

where d_j denotes the number of Γ–variables y_i of degree j in Y and $n = |y_m|$.

Proof. In the proof we omit the subscript to $\otimes$ when the tensor product is taken over R.
As U is Golod, there is a short exact sequence of complexes of free R–modules:

$$0 \to U \xrightarrow{\xi} U \otimes T(V) \to U \otimes T(V) \otimes V \to 0,$$

where ξ is the obvious inclusion of U into the DG U–module constructed in (1.2). Note that the middle complex $U \otimes T(V)$ is an R–free resolution of k, and that the differential on the last complex is given by the formula $\partial(u \otimes v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes v) = \partial(u \otimes v_{i_1} \otimes \cdots \otimes v_{i_n}) \otimes v$. Thus, taking homology in the exact sequence of complexes:

$$0 \to M \otimes U \xrightarrow{M \otimes \xi} M \otimes U \otimes T(V) \to M \otimes U \otimes T(V) \otimes V \to 0,$$

one obtains an exact sequence of graded k–vector spaces:

$$0 \to C \to \mathrm{H}(M \otimes U) \xrightarrow{\mathrm{H}(M \otimes \xi)} \operatorname{Tor}^R(M, k) \to \operatorname{Tor}^R(M, k) \otimes_k (k \otimes V) \to \mathfrak{s}C \to 0,$$

where $\mathfrak{s}C$ is obtained from C by degree-shifting: $(\mathfrak{s}C)_i = C_{i-1}$. As $|\mathfrak{s}C|(t) = t \cdot |C|(t)$ and $|k \otimes V|(t) = t \cdot (|\mathrm{H}(U)|(t) - 1)$, the homology sequence provides an equality:

$$P_M^R(t) = \frac{|\mathrm{H}(M \otimes U)|(t) - (1+t) \cdot |C|(t)}{1 + t - t \cdot |\mathrm{H}(U)|(t)}.$$

We want to prove that all three series which appear in the fraction are rational, with $q(t) = \prod_{0 \leqslant 2i \leqslant n} (1 - t^{2i})^{d_{2i}}$ as denominator. For $|\mathrm{H}(U)|(t)$ and $|\mathrm{H}(M \otimes U)|(t)$ it suffices to apply Gulliksen's result [22], described in (1.3).

The same result shows that the corresponding conclusion for $|C|(t)$ will follow if we prove that C is a $\mathcal{T}$–submodule of $\mathrm{H}(M \otimes U)$. To this end, let $U\langle X\rangle$ be an acyclic closure of U, and consider a comparison $\chi : U \otimes T(V) \to U\langle X\rangle$ of R–free resolutions of k. It is a homotopy equivalence. As both the inclusion $\zeta : U \to U\langle X\rangle$ and the composition $\chi\xi$ are homomorphisms from a free complex to a resolution and induce the same map on $\mathrm{H}_0(U)$, they are homotopic. This yields a homotopy from $(M \otimes \chi) \circ (M \otimes \xi)$ to $M \otimes \zeta$. In the resulting equality:

$$\mathrm{H}(M \otimes \chi) \circ \mathrm{H}(M \otimes \xi) = \mathrm{H}(M \otimes \zeta)$$

the map $\mathrm{H}(M \otimes \chi)$ is bijective, hence we get $C = \mathrm{Ker}\,(\mathrm{H}(M \otimes \xi)) = \mathrm{Ker}\,(\mathrm{H}(M \otimes \zeta))$. By (1.3.5) there is an action of $\mathcal{T}$ on $M \otimes U\langle X\rangle$, for which $M \otimes \zeta$ is a homomorphism of DG $\mathcal{T}$–modules. The kernel of the induced homomorphism $\mathrm{H}(M \otimes \zeta)$ is then a $\mathcal{T}$–submodule of $\mathrm{H}(M \otimes U)$, as desired.

To complete the proof of the first assertion of the theorem, just multiply with $q(t)$ both the numerator and the denominator of the expression for $P_M^R(t)$, and set:

$$\mathrm{Den}^R(t) = q(t) \cdot (1 + t - t \cdot |\,\mathrm{H}(U)|(t))\,.$$

Finally, let $M = k$. As $\partial(U) \subseteq \mathfrak{m}_R U$, the vector space $k \otimes U$ has a trivial differential, hence

$$|H(k \otimes U)|(t) = |k \otimes U|(t) = \frac{\prod_{0 \leqslant 2i+1 \leqslant n} (1 + t^{2i+1})^{d_{2i+1}}}{q(t)}\,.$$

Thus, in order to show that $P_k^R(t)$ has the form prescribed by the theorem, we have to show that $|C|(t) = 0$, that is, that the inclusion $k \otimes \zeta : k \otimes U \hookrightarrow k \otimes U\langle X\rangle$ induces an injective map in homology. This is clear, since by (1.3.4) the complex $k \otimes U\langle X\rangle$ has a trivial differential as well. □

(1.5) Partial acyclic closures. Let $R\langle X\rangle$ be an acyclic closure of R, with the Γ–variables ordered in such a way that $|x_i| \le |x_j|$ when $i < j$. The number of Γ–variables of degree n in an acyclic closure of R is equal to the n'th deviation $e_n(R)$. Note that $e = e_1(R)$ is the embedding dimension of R, and that $R\langle x_1, \dots, x_e\rangle$ is the Koszul complex on a minimal set of generators of $\mathfrak{m}$. If $m \ge e$, then $U = R\langle x_1, \dots, x_m\rangle$ is a distinguished DG R–algebra: In this situation (1.3.4) is Gulliksen's theorem [24, (1.6.2)] on the minimality of acyclic closures, while (1.3.3) is an important step in its proof.

Thus, Theorem (1.4) has the following consequence:

Theorem. *Let R be a local ring with deviations $e_i(R) = e_i$ and acyclic closure $R\langle X\rangle = R\langle x_1, x_2, \dots\rangle$, where the Γ–variables satisfy the condition that $|x_i| \le |x_j|$ for $i < j$.*

If for some integer $m \ge e_1$ the DG R–algebra $R\langle x_1, \dots, x_m\rangle$ is Golod, then there is a polynomial $\mathrm{Den}^R(t) \in \mathbb{Z}[t]$ and for any finitely generated R–module M there is a polynomial $p_M(t) \in \mathbb{Z}[t]$ such that:

$$P_M^R(t) = \frac{p_M(t)}{\mathrm{Den}^R(t)}\,.$$

Furthermore, there are equalities:

$$p_k(t) = \begin{cases} (1+t)^{e_1}(1+t^3)^{e_3} \;\cdots\; (1+t^n)^r \;\; \textit{with } r = m - \sum_{i=1}^{n-1} e_i & \textit{if } \; n = |x_m| \;\; \textit{is odd}; \\ (1+t)^{e_1}(1+t^3)^{e_3} \;\cdots\; (1+t^{n-1})^{e_{n-1}} & \textit{if } \; n = |x_m| \;\; \textit{is even}. \end{cases}$$

□

(1.6) Formal local rings. Let $\widehat{R}$ denote the $\mathfrak{m}_R$–adic completion of the local ring R. By Cohen's Structure Theorem, $\widehat{R}$ is a homomorphic image of a complete regular local ring. We say that R is *formal* if for some Cohen presentation $R \cong Q/\mathfrak{a}$ with Q complete regular, a minimal free resolution of the Q–module $\widehat{R}$ has a structure of DG Q–algebra.

The motivation behind the name comes from the dictionary with rational homotopy theory, cf. [5] and [11]. In the latter environment, a *formal space* may be defined by the condition that its Sullivan-DeRham commutative cochain algebra is isomorphic with its rational cohomology algebra in the derived category of the category of one-connected DG $\mathbb{Q}$–algebras. It will be shown in the proof of the next result that if R is formal, then the Koszul complex on a minimal set of generators of $\mathfrak{m}_R$ is isomorphic with its own homology algebra in the derived category of the category of DG R–algebras.

Theorem. *Let R be a local ring of embedding dimension $e_1 = e$, and let $K = R\langle x_1, \dots, x_e\rangle$ be the Koszul complex on a minimal set of generators of $\mathfrak{m}_R$. Let $T\langle Y\rangle = T\langle y_{e+1}, y_{e+2}, \dots\rangle$ be an acyclic closure of the graded k–algebra $T = \mathrm{H}(K)$, with $|y_i| \le |y_j|$ for $i < j$.*

If R is formal, and for some integer $m \ge e$ the DG k–algebra $T\langle y_{e+1}, \dots, y_m\rangle$ is Golod, then $\widehat{R}$ has an acyclic closure $\widehat{R}\langle x_1, \dots, x_e, x_{e+1}, \dots\rangle$ such that $|x_i| = |y_i|$ for $i > e$ and $\widehat{R}\langle x_1, \dots, x_m\rangle$ is Golod.

Corollary. *$P^R_M(t)$ and $p_k(t)$ are given by the formulas in* (1.5) *with $n = |y_m|$.*

Proof of Theorem. Note that $\widehat{R} \otimes_R K$ is the Koszul complex on a minimal set of generators of $\mathfrak{m}_{\widehat{R}}$, and its homology is canonically isomorphic to T. Thus we may assume R is complete, and write $R = Q/\mathfrak{a}$ with Q regular.

Next we show that if there is a $t \in \mathfrak{a}$ with $t \notin \mathfrak{m}_Q^2$, then the minimal free resolution of R over the regular local ring $Q' = Q/(t)$ admits a DG Q'–algebra structure. Such a t can be extended to a minimal system of generators of $\mathfrak{m}_Q$. Let $E = Q\langle x'_1, \dots x'_f\rangle$ be the Koszul complex on such a system, with $\partial(x'_f) = t$, and note that E is an acyclic closure of Q. Let G be a minimal algebra resolution of Q over R. Choose $g \in G_1$ such that $\partial(g) = t$ and note that g can be extended to a basis of G_1. Setting $x_i = 1 \otimes x'_i$, we get surjective quasi-isomorphisms:

$$R\langle x_1, \dots, x_f\rangle = R \otimes_Q E \leftarrow G \otimes_Q E \to G \otimes_Q k = \overline{G}$$

(we use the term *quasi-isomorphism* to denote a homomorphism of DG algebras which induces an isomorphism in homology).

Because of $\partial(x_f) = 0$, we have $\mathrm{H}(R\langle x_1, \dots x_f\rangle) = \mathrm{H}(R\langle x_1, \dots x_{f-1}\rangle)\langle x_f\rangle$. Under the quasi-isomorphisms above, the class of the cycle $-x_f$ on the left-hand side corresponds to that of $\overline{g} = g \otimes 1$ on the right-hand side. It follows that $\overline{G} = \mathrm{H}(\overline{G})$ is a free graded module over $k\langle \overline{g}\rangle$. Let $\{ g_s \in G \}_{s \in S}$ be a set of homogeneous elements of G whose image $\{ \overline{g}_s \in \overline{G} \}_{s \in S}$ is a basis of $\overline{G}$ over $k\langle \overline{g}\rangle$. As the Q–algebra $G^\natural$ is graded commutative, and $|g| = 1$, we have $g^2 = 0$, and thus $Q\langle g\rangle^\natural$ is a graded ring with Jacobson radical $\mathfrak{m}_Q \oplus Qg$. Nakayama's Lemma implies that $\{ g_s \}_{s \in S}$ generates $G^\natural$ as a graded $Q\langle g\rangle^\natural$–module, hence $G^\natural$ is a homomorphic image of a free graded $Q\langle g\rangle^\natural$–module with basis $\{ c_s \mid |c_s| = |g_s| \}_{s \in S}$.

Since both are free modules with equal ranks over Q, they are isomorphic, and we conclude that the graded $Q\langle g\rangle^\natural$–module $G^\natural$ is free on $\{ g_s \}_{s\in S}$.

Note that $Q\langle g\rangle$ is a DG subalgebra of G, and that both are graded by non-negative integers. It follows that G, viewed as a DG module over $Q\langle g\rangle$, is semi-free in the sense of [11, (1.11)]. This implies that the quasi-isomorphism $\rho : Q\langle g\rangle \to Q/(t) = Q'$, which is the natural map on Q and sends g to 0, induces a quasi-isomorphism:

$$G = Q\langle g\rangle \otimes_{Q\langle g\rangle} G \xrightarrow{\rho\otimes G} Q' \otimes_{Q\langle g\rangle} G = G' .$$

As a consequence, $\mathrm{H}(G') = R$. Furthermore, $\{ 1\otimes g_s \mid s \in S \ , \ |g_s| = i \}$ is a basis for the Q'–module G'_i, and $\partial(G'_i) \subseteq \mathfrak{m}_Q(G'_{i-1}) = \mathfrak{m}_{Q'}(G'_{i-1})$, so that the DG Q'–algebra G' is a minimal free resolution of the Q'–module R.

After iterating this procedure, and once more changing notation, for the rest of the proof we assume $R = Q/\mathfrak{a}$ with $\mathfrak{a} \subseteq \mathfrak{m}_Q^2$, and fix a minimal DG Q–algebra resolution G of R. In the quasi-isomorphisms considered at the beginning of this proof we now have $f = e$, hence we get an induced isomorphism of graded k–algebras $T \cong \mathrm{H}(\overline{G}) = \overline{G}$. This yields a diagram of surjective quasi-isomorphisms:

$$K \xleftarrow{\alpha} G \otimes_Q E \xrightarrow{\beta} T .$$

By induction we can assume that for some $j \geq 0$ it has been embedded into a diagram of surjective quasi-isomorphisms:

$$K\langle x_{e+1},\dots,x_{e+j}\rangle \xleftarrow{\alpha^{(j)}} (G\otimes_Q E)\langle y'_{e+1},\dots,y'_{e+j}\rangle = S^{(j)} \xrightarrow{\beta^{(j)}} T\langle y_{e+1},\dots,y_{e+j}\rangle = T^{(j)}$$

where $\alpha^{(j)}$ (respectively, $\beta^{(j)}$) maps each Γ–variable y'_i to the Γ–variable x_i (respectively, y_i); furthermore, if $|y'_i|$ is even, then its divided powers are sent to the corresponding ones of x_i (respectively, y_i).

Surjective quasi-isomorphisms are surjective on cycles, so there is a cycle $z'_{e+j+1} \in S^{(j)}$ with $\beta^{(j)}(z'_{e+j+1}) = \partial(y_{e+j+1}) \in T^{(j)}$. Adjoin Γ–variables y'_{e+j+1} to $S^{(j)}$ and x_{e+j+1} to $K\langle x_{e+1},\dots,x_{e+j}\rangle$, killing z'_{e+j+1} and $\alpha^{(j)}(z'_{e+j+1})$, respectively. The maps $\alpha^{(j)}$ and $\beta^{(j)}$ can be extended to homomorphisms of DG Q–algebras

$$K\langle x_{e+1},\dots,x_{e+j}\rangle\langle x_{e+j+1}\rangle \xleftarrow{\alpha^{(j+1)}} S^{(j)}\langle y'_{e+j+1}\rangle \xrightarrow{\beta^{(j+1)}} T^{(j)}\langle y_{e+j+1}\rangle$$

such that $\alpha^{(j+1)}(y'_{e+j+1}) = x_{e+j+1}$ and $\beta^{(j+1)}(y'_{e+j+1}) = y_{e+j+1}$ – with the usual provision for divided powers. By construction, these are surjective homomorphisms of DG Q–algebras, and it is easy to see that they induce isomorphisms in homology, cf. e.g. [24, (1.3.5)]. In the limit, they yield surjective quasi-isomorphisms:

$$R\langle X\rangle = K\langle x_{e+1}, x_{e+2}\dots\rangle \xleftarrow{\alpha} (G\otimes_Q E)\langle y'_{e+1}, y'_{e+2},\dots\rangle \xrightarrow{\beta} T\langle y_{e+1}, y_{e+2},\dots\rangle$$

so that the homology of all three DG algebras is equal to k.

Note that for $n \geq 2$ the set $\partial(Y_n)$ is a basis of $\mathrm{H}_{n-1}(T\langle Y_{\leq n}\rangle)$, because $T\langle Y\rangle$ is an acyclic closure of T. The quasi-isomorphisms already in place show that $\partial(X_n)$ is a basis of $\mathrm{H}_{n-1}(R\langle X_{\leq n}\rangle)$ for $n \geq 2$. Thus, $R\langle X\rangle$ is an acyclic closure of the Koszul complex $R\langle x_1, \ldots, x_e\rangle$, and this means it is one of R as well. In order to finish the proof we have to show that $R\langle x_1, \ldots, x_m\rangle$ is Golod. To this end it suffices to prove that the trivial Massey operation given on $T\langle y_1, \ldots, y_m\rangle$ passes through the surjective quasi-isomorphisms $\beta^{(m)}$ and $\alpha^{(m)}$.

This is exactly what the argument starting on the bottom of p. 597 of [4] does. □

Proof of Corollary. Koszul homology and Betti numbers being invariant under completion, it suffices to establish the formulas for $\widehat{R}$ and $\widehat{R} \otimes_R M$ in place of R and M, respectively. In this case the preceding Theorem shows that Theorem (1.5) applies. □

(1.7) Generalized Golod rings. Let U be a DGΓ R–algebra, that is, a DG R–algebra which has a system of divided power operations satisfying the usual conditions, cf. e.g. [24]. The Eilenberg-Moore derived functor $\mathrm{Ext}^*_U(k\,,k)$, equipped with the Yoneda product structure, then contains a canonically defined graded k–subspace $\pi^*(U)$ which is closed under the formation of graded commutators and of squares of elements of odd degree. Thus, $\pi^*(U)$ is a graded Lie algebra, called the *homotopy Lie algebra* of U, cf. [5]. The assignment $U \mapsto \pi^*(U)$ provides a contravariant functor from the category of DGΓ R–algebras to that of graded Lie k–algebras. A graded Lie algebra over a field k is *free* if its universal enveloping algebra is free as an associative k–algebra; such Lie algebras are free objects in the category of graded Lie algebras, cf. [34].

We say that a local ring R is *generalized Golod* (respectively, *of level* $\leq n$) if $\pi^*(R)$ contains a free graded Lie subalgebra of finite codimension (respectively, if the subalgebra $\pi^{>n}(R)$ is free). Interesting cohomological properties of generalized Golod rings R of level n can be deduced from [35]: The Yoneda algebra $\mathrm{Ext}^*_R(k\,,k)$ is coherent (cf. [*idem*, Theorem 3]) and its finitistic global dimension is at most $\sum_{1\leqslant 2i\leqslant n} e_{2i}$ (cf. [*idem*, Theorem 2]).

Since subalgebras of free graded Lie algebras are free [30, (A.1.10)], any generalized Golod ring R is generalized Golod of level $\leq n$ for some $n \in \mathbb{N}$, and then it is generalized Golod of level $\leq q$ for any $q \geq n$. The connections with Golod DG R–algebras is as follows:

Lemma. *Let R be a local ring with acyclic closure $R\langle X\rangle$.*

If $R\langle X\rangle$ has a distinguished Golod DG subalgebra U, then R is generalized Golod.

The ring R is generalized Golod of level $\leq n$ if and only if the DG R–algebra $R\langle X_{\leq n}\rangle$ is Golod.

Proof. It follows from condition (1.3.2) that a distinguished DG R–algebra is a DGΓ R–algebra. By [6, (3.4)] a DGΓ algebra U is Golod if and only if its homotopy Lie algebra $\pi^*(U)$ is free. Thus, in view of the remarks above both assertions of the Lemma follow from the fact that the inclusion of DG R–algebras $\eta : R \hookrightarrow U = R\langle X_{\leq n}\rangle$ induces an injective homomorphism of graded Lie algebras $\pi^*(\eta) : \pi^*(U) \to \pi^*(R)$, with image equal to $\pi^{>n}(R)$ when $U = R\langle X_{\leq n}\rangle$.

As $R\langle X\rangle$ is a semi-free resolution of k over R and over U, the homomorphism $\pi^*(\eta)$ is the dual of the map induced on Γ–indecomposables by the homomorphism of DGΓ k–algebras:

$$\mathrm{H}(k\otimes_\eta R\langle X\rangle):\mathrm{H}(k\otimes_R R\langle X\rangle)\to\mathrm{H}(k\otimes_U R\langle X\rangle)\,,$$

cf. [5, §2] or [11, §5]. Due to the inclusions $\partial(U\langle X\rangle)\subseteq\mathfrak{m}_R(U\langle X\rangle)\subseteq\mathfrak{m}_U(U\langle X\rangle)$, both complexes have trivial differentials, and thus:

$$\mathrm{H}(k\otimes_\eta R\langle X\rangle)=k\otimes_\eta R\langle X\rangle:k\otimes_R R\langle X\rangle\to k\otimes_U R\langle X\rangle\,.$$

This homomorphism of DGΓ k–algebras is surjective, and hence so is the homomorphism induced on their spaces of Γ–indecomposables.

If $U=R\langle X_{\leq n}\rangle$, then clearly the map on indecomposables is trivial in degrees $\leq n$ and bijective in degrees $>n$. □

(1.8) Applications.

(1.8.1) *Golod rings.* The original definition of *Golod ring* in [24] is that the *Koszul complex* $K=R\langle X_1\rangle$ is a DG R–algebra which admits a trivial Massey operation. Thus, the generalized Golod rings of level ≤ 1 are precisely the rings studied first by Golod [20], who has characterized them by the equality:

$$P^R_{R/\mathfrak{m}}(t)=\frac{(1+t)^e}{1-\sum_{i=1}^e\dim_k\mathrm{H}_i(K)t^{i+1}}\,.$$

In this case, Theorem A gives Ghione and Gulliksen's result [19, Theorems 4 and 1] that each $P^R_M(t)$ is rational with denominator $1-\sum_{i=1}^e\dim_k\mathrm{H}_i(K)t^{i+1}$.

(1.8.2) *Complete intersections.* Tate [41, Theorem 6] has shown that an acyclic closure of a codimension c complete intersection R is obtained by adjunction of variables in degrees 1 and 2 only, that is, has the form $R\langle X\rangle$ with $X=X_{\leq 2}$. Thus,

$$P^R_{R/\mathfrak{m}}(t)=\frac{(1+t)^e}{(1-t^2)^c}=\frac{(1+t)^{e-c}}{(1-t)^c}\,.$$

and $\pi^*(R)$ is concentrated in degrees ≤ 2; in particular R is Golod of level ≤ 2.

Theorem A applied to R shows that each $P^R_M(t)$ is rational with denominator $(1-t^2)^c$, a fact first established by Gulliksen in [21, (4.1.ii)].

(1.8.3) *Complete-intersection-by-Golod rings.* If the completion of a local ring R is a homomorphic image of a complete intersection C by a Golod homomorphism, then by [2, (3.5.4)] or [32, Corollary (2.4)], the canonical homomorphism $\pi^*(R)\to\pi^*(C)$ is surjective with kernel a free Lie algebra. Thus, R is a Golod ring of level ≤ 2.

In this case Theorem A coincides with Levin's result, cf. [12, (5.18)].

(1.8.4) *Generalized Golod rings of higher level.* More than 10 years ago, Löfwall [33] has constructed a ring with $\operatorname{edim} R - \operatorname{depth} R = 7$ which is generalized Golod of level 3, and for which the Lie subalgebra of $\pi^*(R)$ generated by $\pi^1(R)$ is nilpotent of class 3. Lately, Kustin and Palmer [29] have shown that rings with such properties appear in the classification of almost complete intersections of codimension 4. Even more recently, a detailed investigation by Roos [36] of the homotopy Lie algebras of graded rings with 4 generators and quadratic relations has turned out several types of generalized Golod rings of level 3, some of them having an infinite dimensional subalgebra generated by $\pi^1(R)$.

While by [12] rings with $\operatorname{edim} R - \operatorname{depth} R \leq 3$ are generalized Golod of level ≤ 2, and by [31, §7] (or by combining Anick's example [1] with Theorem A) there are rings with $\operatorname{edim} R - \operatorname{depth} R \geq 5$ which are not generalized Golod, I do not know whether such rings exist when $\operatorname{edim} R - \operatorname{depth} R = 4$.

2. Special complete intersections

In this section we investigate the polynomial character of Betti functions of modules over a complete intersection R. For the proofs, we use the fact that a Betti function is eventually polynomial if and only if the corresponding Poincaré series is rational with denominator a power of $(1-t)$. Since by Gulliksen's result Poincaré series over a complete intersection are rational with poles at $t = \pm 1$ only, cf. (1.8.2), the polynomial condition on b_M^R means that $P_M^R(t)$ is regular at $t = -1$. However, the algebraic significance of this pole remains unclear, in sharp contrast with that at $t = 1$, whose order carries important structural information on M, cf. [7].

(2.1) Poincaré series of powers of the maximal ideal. When R is a complete intersection, the Poincaré series of its maximal ideal $\mathfrak{m}$ depends only on the embedding dimension e and the codimension c, as demonstrated by Tate's formula recalled in (1.8.2). For higher powers of $\mathfrak{m}$, the corresponding expressions contain more data on the singularity of the ring. This is demonstrated by the result for $\mathfrak{m}^2$, which involves the number of quadratic relations $q(R) = \binom{e+1}{2} - \dim_k \mathfrak{m}^2/\mathfrak{m}^3$. We note that if the defining relations f_h are all contained in $\mathfrak{m}^r$ for some $r \geq 2$, then an argument similar to the one below can be used to compute $P^R_{R/\mathfrak{m}^i}(t)$ for $1 \leq i \leq r$; details are omitted.

Theorem. *If $(R, \mathfrak{m})$ is a local complete intersection of embedding dimension e, codimension c, and with q quadratic relations, then:*

$$P^R_{R/\mathfrak{m}^2}(t) = \frac{(1-t)^q + (1+t)^{e-q-1} \cdot (et-1)}{t \cdot (1+t)^{c-q-1} \cdot (1-t)^c}\,.$$

Proof. Clearly, we may assume that R is complete, and hence that $R = Q/(f_1, \ldots, f_c)$ for some regular local ring Q and a Q–regular sequence $f_1, \ldots, f_c$ contained in the square of the maximal ideal $\mathfrak{n}$ of Q. Let $t_1, \ldots, t_e$ be a minimal system of generators of $\mathfrak{n}$, and let $E = Q\langle u_1, \ldots, u_c\rangle$ be the Koszul complex with $\partial(u_h) = f_h$ for $1 \leq h \leq c$. For $M = R/\mathfrak{m}^2 = Q/\mathfrak{n}^2$ we shall use a minimal resolution G, introduced by Buchsbaum and

Eisenbud [15] and studied in detail by Srinivasan, who has shown [39, (3.6)] that G has a structure of DG Q–algebra. Let $\{ y_{ij} \mid 1 \le i \le j \le e\}$ be a basis of G_1 such that $\partial(y_{ij}) = t_i t_j$. As f_h is contained in $\mathfrak{n}^2$ for $1 \le h \le c$, we can write the defining equations of the complete intersection in the form:

$$f_h = \sum_{1 \leqslant i \leqslant j \leqslant e} a_{h,ij} t_i t_j \,, \quad a_{h,ij} \in Q \,.$$

Since E is free as a graded commutative Q–algebras, the homomorphism of free Q–modules $E_1 \to G_1$ which maps u_h to $\sum_{1 \leqslant i \leqslant j \leqslant e} a_{h,ij} y_{ij}$ extends to a a homomorphism of graded algebras $\varphi : E \to G$. For $1 \le h \le c$ we have by construction $\partial\varphi(u_h) = \varphi\partial(u_h)$, hence φ is a homomorphism of DG Q–algebras. We give G the structure of DG E–module defined by φ.

From [3, (3.1.1)] we have a spectral sequence:

$${}^2E_{p,q} = \operatorname{Tor}_p^{\operatorname{Tor}^Q(R\,,k)}\left(\operatorname{Tor}^Q(R/\mathfrak{m}^2\,,k)\,,k\right)_q \Longrightarrow \operatorname{Tor}^R_{p+q}(R/\mathfrak{m}^2\,,k)\,.$$

Furthermore, since the minimal Q–free resolution G of $R/\mathfrak{m}^2$ has a structure of DG module over the minimal Q–free resolution E of R, [3, (4.1.1)] shows that the sequence collapses with ${}^2E = {}^\infty E$. This provides us with an equality of formal power series:

$$P^R_{R/\mathfrak{m}^2}(t) = \sum_{n=0}^{\infty} \left(\sum_{p+q=n} \dim_k \operatorname{Tor}_p^{\operatorname{Tor}^Q(R\,,k)}\left(\operatorname{Tor}^Q(R/\mathfrak{m}^2\,,k)\,,k\right)_q \right) \cdot t^n = P^{\overline{E}}_{\overline{G}}(t)\,,$$

where $\overline{E} = E \otimes_Q k = \operatorname{Tor}^Q(R\,,k)$ and $\overline{G} = G \otimes_Q k = \operatorname{Tor}^Q(R/\mathfrak{m}^2\,,k)$.

Of course, $\overline{E}$ is the exterior algebra on a c–dimensional vector space over k. On the other hand, it is proved in [39, (3.4)] that the DG Q–algebra G satisfies $G_+^2 \subset (t_1, \dots, t_e)G$, and this implies that $\overline{G}$ is a graded k–algebra with $\overline{G}_+^2 = 0$. Note that q is the rank over k of the $c \times \binom{e+1}{2}$ matrix $(\overline{a}_{h,ij})$, where bars denote reductions modulo $\mathfrak{n}$. Thus, we can choose a basis $e_1, \dots, e_c$ of $\overline{E}_1$, such that $e_{q+1}, \dots, e_c$ is a basis of $\operatorname{Ker}(\varphi_1 \otimes k)$. For the rest of the proof we identify $\overline{E}$ with $A \otimes_k B$, where A and B are exterior algebras generated by $e_1, \dots, e_q$ and $e_{q+1}, \dots, e_c$, respectively. The preceding discussion then shows that there are isomorphisms of graded $\overline{E}$–modules:

$$\overline{G} \cong \left(\overline{E}/(A_+^2 + B_+)\overline{E}\right) \oplus W \cong \left(A/A_+^2 \otimes_k B/B_+\right) \oplus W\,,$$

where W is a graded k–vector space such that $(\overline{E}_+)W = 0$.

In terms of Poincaré series, this yields an equality:

$$P^{\overline{E}}_{\overline{G}}(t) = P^A_{A/A_+^2}(t) \cdot P^B_k(t) + P^{\overline{E}}_k(t) \cdot |W|(t)\,.$$

In order to compute the Hilbert series of W, recall that by construction of the complex G the ranks of its modules are given by $\operatorname{rank}_Q G_0 = 1$ and $\operatorname{rank}_Q G_i = i\binom{e+1}{i+1}$ for $1 \le i \le e$.

Since W is the quotient of $\overline{G}$ by the subspaces spanned by $1 \in \overline{G}_0$ and the images in $\overline{G}_1$ of $e_1, \dots, e_q$, we get:

$$|W|(t) = \frac{1 - qt^2 + (1+t)^e \cdot (et - 1)}{t}\,.$$

It remains to compute three Poincaré series over the exterior algebras A, B, and $\overline{E}$. For any a and b such that $1 \le a < b \le c$, the DG k–algebra $k\langle e_a, \dots, e_b\rangle\langle v_a, \dots, v_b\rangle$, with $\partial(v_i) = e_i$ for $a \le i \le b$, is well known and easily seen to be an acyclic closure of the exterior algebra $k\langle e_a, \dots, e_b\rangle$. Thus, for two of these series we get immediately:

$$P_k^B(t) = \frac{1}{(1-t^2)^{c-q}} \quad \text{and} \quad P_k^{\overline{E}}(t) = \frac{1}{(1-t^2)^c}\,.$$

Using such a resolution to obtain $\operatorname{Tor}_i^A(A/A_+^2\,, k)_j$, we see that for $i \ge 1$ this group is trivial unless $j = i+1$, and then there is an exact sequence:

$$0 \to \Gamma_{i+1} \to A_1 \otimes_k \Gamma_i \to \operatorname{Tor}_i^A(A/A_+^2\,, k)_{i+1} \to 0\,,$$

where $\Gamma_i = (k\langle v_1, \dots, v_q\rangle)_{2i}$. It follows that $\dim_k \operatorname{Tor}_i^A(A/A_+^2\,, k)_{i+1} = i\binom{q+i-1}{i+1}$ for $i \ge 1$, and a computation with formal power series then yields:

$$P_{A/A_+^2}^A(t) = \frac{-1 + qt^2 + (1+t)\cdot(1-t^2)^q}{t \cdot (1-t^2)^q}\,.$$

The desired expression for $P_{R/\mathfrak{m}^2}^R(t)$ is obtained by putting together the results of the preceding computations. □

(2.2) An example of group representations. The difference of the behavior of group cohomology at $p = 2$ and at odd p is well known in modular representation theory. The preceding theorem gives an interesting illustration of this phenomenon.

When k be a field of characteristic $p > 0$ the matrices:

$$g_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

define a representation of an elementary abelian p–group of rank two, G, in a 3–dimensional k–vector space, V. Under the isomorphism of k–algebras $k[G] \cong k[[t_1, t_2]]/(t_1^p, t_2^p) = R$, which maps g_i to the class of $(1 - t_i)$ for $i = 1, 2$, the representation V corresponds to the R–module $M = R/(t_1, t_2)^2$, and there are isomorphisms: $\mathrm{H}_n(G, V) \cong \operatorname{Tor}_n^R(M\,, k)$.

If p is odd, then (2.1) yields $P_V^G(t) = (1+2t)/(1-t^2)(1+t)$, which gives two distinct polynomials for the odd and the even Betti numbers:

$$b_n^G(V) = \begin{cases} \frac{3}{2}n + 1 & \text{for } \textit{even} \;\; n \ge 0\,; \\ \frac{3}{2}n + \frac{3}{2} & \text{for } \textit{odd} \;\; n \ge 1\,. \end{cases}$$

If $p = 2$, then (2.1) shows that $P_V^G(t) = (1 - t + t^2)/(1-t)^2$, hence:

$$b_n^G(V) = n \quad \text{for } n \ge 1\,.$$

The particularly simple form of the Poincaré series in the second case belongs to a general pattern described in the next result.

(2.3) Complete intersections of minimal multiplicity. As noted in the Introduction, the multiplicity of a codimension c complete intersection is at least 2^c.

Theorem. *If $(R,\mathfrak{m})$ is a codimension c complete intersection of multiplicity 2^c, then for any finitely generated R–module M there is a polynomial $q_M(t)\in\mathbb{Z}[t]$ such that:*

$$P_M^R(t)=\frac{q_M(t)}{(1-t)^c}\,.$$

Proof. There is no loss of generality in assuming that R is complete and that the residue field $k=R/\mathfrak{m}$ is infinite. Let $R=Q/(f_1,\dots,f_c)$ be a Cohen presentation of R with Q a regular local ring whose maximal ideal $\mathfrak{n}$ is minimally generated by $t_1,\dots,t_e$, and in which

$$f_h=\sum_{1\leqslant i\leqslant j\leqslant e} a_{h,ij}t_it_j \quad \text{with } a_{h,ij}\in Q \text{ for } 1\le h\le c\,.$$

The associated graded ring $A=\operatorname{gr}_{\mathfrak{n}}(Q)$ is a polynomial k–algebra in the initial forms $t_1^*,\dots,t_c^*$. By [37, (1.8)] the assumption of minimal multiplicity implies that the initial forms $f_1^*,\dots,f_c^*$ form an A–regular sequence, such that $B=\operatorname{gr}_{\mathfrak{m}}(R)$ is isomorphic to $A/(f_1^*,\dots,f_c^*)A$. Thus, B is a graded complete intersection with quadratic relations

$$f_h^*=\sum_{1\leqslant i\leqslant j\leqslant e}\overline{a}_{h,ij}t_i^*t_j^*\,,\quad 1\le h\le c\,.$$

As k is infinite, one can choose in $\mathfrak{m}\smallsetminus\mathfrak{m}^2$ a sequence of elements $\mathbf{x}=x_1,\dots,x_{e-c}$ whose initial forms constitute a B–regular sequence. It follows from [14, §7 Proposition 7], cf. also [37, (1.4.ii)] that the multiplicity of $R/(\mathbf{x})R$ is equal to that of R. On the other hand, $\mathbf{x}$ is an R–regular sequence, and hence is also regular on any maximal Cohen–Macaulay R–module, in particular on the $(e-c)$'th syzygy N of M. As for any $n\ge 0$ there are equalities $b_n^{R/(\mathbf{x})R}(N/(\mathbf{x})N)=b_n^R(N)=b_{n+e-c}^R(M)$, for the rest of the proof we may assume R is artinian with embedding dimension and codimension both equal to c.

Recall that the universal enveloping algebra of $\pi^*(R)$ is $\operatorname{Ext}_R^*(k\,,k)$ with the Yoneda product structure, and that $\operatorname{Ext}_R^*(M\,,k)$ is a graded (left) module for this structure. By [21, (3.1)], cf. also [7, (2.1)], $\operatorname{Ext}_R^*(M\,,k)$ is a finitely generated module over a graded polynomial k–algebra on c variables of degree 2. In [7, (6.1.2)] this algebra is identified with the subalgebra of $\operatorname{Ext}_R^*(k\,,k)$ generated by $\pi^2(R)$, which is central in $\operatorname{Ext}_R^*(k\,,k)$. Thus, $\operatorname{Ext}_R^*(M\,,k)$ is finitely generated over $\operatorname{Ext}_R^*(k\,,k)$. The Lie algebra $\pi^*(R)$ is concentrated in degrees 1 and 2, and Sjödin [38, §3] has shown that the bracket and quadratic operator from $\pi^1(R)$ to $\pi^2(R)$ are completely determined by the matrix $(\overline{a}_{h,ij})$, where bars denote reduction modulo $\mathfrak{n}$.

Applying the graded version of Sjödin's result to the ring B we see that $\pi^*(R)\cong\pi^*(B)$ as graded Lie algebras, and hence the graded k–algebras $\operatorname{Ext}_R^*(k\,,k)$ and $\operatorname{Ext}_B^*(k\,,k)=\mathcal{E}$ are isomorphic as well. In order to finish the proof, we shall show that for any finitely generated graded (left) $\mathcal{E}$–module $\mathcal{M}$ there is a polynomial $q_{\mathcal{M}}(t)\in\mathbb{Z}[t]$ such that:

$$|\mathcal{M}|(t)=\frac{q_{\mathcal{M}}(t)}{(1-t)^c}\,.$$

By Tate's formula recalled in (1.8.2), this holds for $\mathcal{M} = \mathcal{E}$ with $q_{\mathcal{E}}(t) = 1$. The usual Euler characteristic argument then yields the desired conclusion for any $\mathcal{E}$–module with a finite free resolution. Being the universal enveloping algebra of a finite-dimensional graded Lie algebra, $\mathcal{E}$ is (left) noetherian, graded, with $\mathcal{E}_0 = k$. Thus, if we show that $\operatorname{Tor}_n^{\mathcal{E}}(k\,,k) = 0$ for $n > c$, it will follow from [12, §8, Corollaires 2 et 5] that any finitely generated $\mathcal{E}$–module has a finite free resolution of length $\le c$. For the graded k–algebra $\mathcal{E}$, there are vector space isomorphisms $\operatorname{Ext}_{\mathcal{E}}^n(k\,,k) \cong \operatorname{Hom}_k(\operatorname{Tor}_n^{\mathcal{E}}(k\,,k)\,,k)$ for $n \in \mathbb{Z}$. Furthermore, being a complete intersection of quadrics, the graded k–algebra B is a homogeneous Koszul algebra, hence there is an isomorphism of graded k–algebras $\operatorname{Ext}_{\mathcal{E}}^*(k\,,k) \cong B$, cf. [28] or [16, Proposition (1.iv)].

This yields $\operatorname{Ext}_{\mathcal{E}}^n(k\,,k) = 0$ for $n > c$, and thus finishes the proof of the theorem. □

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