COHOMOLOGY OPERATORS DEFINED BY A DEFORMATION

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Introduction

A lot of (co)homological information on modules over a commutative ring R is encoded in terms of composition products of various Ext and Tor modules. Two main difficulties in using this information are that the resulting algebra and module structures are seldom finite, and the products are almost never commutative. One significant exception occurs when $R = Q/(\boldsymbol{x})$ for a Koszul-regular set $\boldsymbol{x} = \{x_1, \dots, x_c\}$ in a commutative ring Q; we think of Q as a deformation of R over a regular base.

Indeed, Gulliksen [8] then constructs a set of commuting operators $\{X_1, \dots, X_c\}$ acting on $\operatorname{Ext}^*_R(M,N)$ by *increasing* degrees by 2 and on $\operatorname{Tor}^R_*(M,N)$ by *decreasing* degrees by 2, making Ext and Tor graded modules over a polynomial ring $\mathcal{S} = R[X_1, \dots, X_c]$ with variables of *cohomological* degree 2. He proves that if $\operatorname{Ext}^n_Q(M,N)$ is noetherian over Q for each n and vanishes for $n \gg 0$, then the graded $\mathcal{S}$–module $\operatorname{Ext}^*_R(M,N)$ is noetherian. This partly overcomes the first obstacle referred to above.

Under more restrictive conditions on Q and $\boldsymbol{x}$, Mehta [9] interprets Gulliksen's operators as composition products: this makes the second difficulty manageable, as the actions of $\operatorname{Ext}^*_R(M,M)$ and $\operatorname{Ext}^*_R(N,N)$ on $\operatorname{Ext}^*_R(M,N)$ are "essentially central."

These results have provided the basis for extensive studies of the (co)homological properties of modules over local complete intersections by Eisenbud [7], and of more general classes of modules of infinite projective dimension by Avramov [1].

Each of the papers quoted above also uses alternative construction(s) of cohomology operators. It has been repeatedly stated in the literature that they yield the same result, but a close inspection of the arguments supporting such claims has revealed serious defects. On the other hand, no single approach seems to give all the essential properties of the cohomology operators which have been extensively used in [4].

In an attempt to provide complete proofs for the coincidence of the various operators, we were led to two new constructions. Both of them come from viewing R–modules as DG (=differential graded) modules over a Koszul complex K resolving R over Q, and our arguments make a full-fledged use of techniques of DG homological algebra, summarized in Section 1. As a bonus, we construct the operators and establish their main properties directly for *complexes* of R–modules.

We first introduce operators *à la* Gulliksen, as connecting homomorphisms. Unlike previous approaches, *both* module arguments are involved from the start. This is important in proving that "left" and "right" versions of earlier operators agree—at least up to sign—which suffices for the applications in [7], [1].

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Our second approach produces cohomology operators from chain-level maps, as did Eisenbud. However, our chain endomorphisms arise not from lifting the differential of a complex of R–modules, but from descending the differential of a DG K–module.

These constructions are presented in Section 2. They are used in Section 3 to provide direct proofs of many formal properties of the cohomology operators, in particular, of their *centrality.* Section 4 compares our approach to earlier ones, filling in gaps and correcting misconceptions concerning the relations between the various operators. Trying to avoid further inaccuracies, we include sufficient detail for most of the essential computations.

In Section 5 we note that some of the existing proofs of the *noetherian nature* of $\operatorname{Ext}^*_R(M,N)$ over the ring of cohomology operators carry over to complexes, and establish a new property of these operators—their *primitivity.* When k is the residue field of a local ring R, we prove that for certain finite R–modules M the graded module $\operatorname{Ext}^*_R(M,k)$ is finite over the subalgebra of $\operatorname{Ext}^*_R(k,k)$ generated by the central and primitive elements of degree 2, even though R itself may not have a deformation. This plays a key role in the investigation of finite CI-dimension in the recent paper [4].

1. Background

This section contains a synopsis of DG homological algebra, taken from [3]. All modules in sight are defined over a fixed commutative ring $\Bbbk$, which is usually suppressed from the notation. In particular, $\otimes$ stands for $\otimes_{\Bbbk}$ and Hom for $\operatorname{Hom}_{\Bbbk}$.

1.1. **Complexes and DG modules.** Differentials have degree -1 and are ubiquitously denoted ∂. The notation $m \in M$ means that m is a homogeneous element of M; we denote its degree by $|m|$. If m is a cycle, then $[m]$ is its homology class. The functor forgetful of differentials is denoted $(-)^\natural$.

Let M and N be complexes. The complex $\operatorname{Hom}(M,N)$ has d'th component $\prod_{i\in\mathbb{Z}}\operatorname{Hom}(M_i,N_{i+d})$, and differential $\partial(\gamma)=\partial\circ\gamma-(-1)^{|\gamma|}\gamma\circ\partial$. A *chain map* γ is a cycle in $\operatorname{Hom}(M,N)$, so that $\partial\circ\gamma=(-1)^{|\gamma|}\gamma\circ\partial$. A chain map of degree 0 is called a *morphism*, and a *quasi-isomorphism* if it induces an isomorphism in homology; the latter is indicated by the appearance of the symbol of $\simeq$ next to its arrow. The complex $M\otimes N$ has d'th component $\coprod_i(M_i\otimes N_{d-i})$ and differential $\partial(m\otimes n)=\partial(m)\otimes n+(-1)^{|m|}m\otimes\partial(n)$.

A commutative DG algebra A is a complex together with a morphism $A\otimes A\to A$ denoted by juxtaposition which has the usual associativity and unit properties, and satisfies $ab=(-1)^{|a||b|}ba$ for all $a,b\in A$ and $a^2=0$ when $|a|$ is odd. An A–module is a complex M, together with a morphism $A\otimes M\to M$, satisfying the usual conditions. If $A_i=0$ for $i\neq 0$, then a DG A–module is simply a complex of modules over the ring A_0, and we call it accordingly.

For A–modules M and N, the subcomplex $\operatorname{Hom}_A(M,N)$ of $\operatorname{Hom}(M,N)$ consists of those γ for which $\gamma(am)=(-1)^{|\gamma||a|}a\gamma(m)$ for all $a\in A$ and $m\in M$; it is an A–module with $(a\gamma)(m)=a(\gamma(m))=(-1)^{|\gamma||a|}\gamma(am)$. Dually, the complex $M\otimes_A N$ is the quotient of $M\otimes N$ by its subcomplex generated by $\{(am)\otimes n-(-1)^{|a||m|}m\otimes(an)\mid a\in A, m\in M, n\in N\}$; it is an A–module, with $a(m\otimes n)=(am)\otimes n=(-1)^{|a||m|}m\otimes(an)$.

1.2. **Resolutions.** An A–module P is *DG-projective* if $\mathrm{Hom}_A(P,-)$ transforms surjective quasi-isomorphisms into surjective quasi-isomorphisms. This is equivalent to the property that $\mathrm{Hom}_A(P,-)$ preserves all quasi-isomorphisms and the graded $A^\natural$–module $P^\natural$ is projective. In that case $(P\otimes_A -)$ preserves quasi-isomorphisms.

An A–module I is *DG-injective* if $\mathrm{Hom}_A(-,I)$ transforms injective quasi-isomorphisms into surjective quasi-isomorphisms. This is equivalent to the properties that $\mathrm{Hom}_A(-,I)$ preserves quasi-isomorphisms and the graded $A^\natural$–module $I^\natural$ is injective.

A bounded below (respectively, above) A–module M such that $M^\natural$ is projective (respectively, injective) over $A^\natural$ is DG-projective (respectively, DG-injective).

Let X be an arbitrary A–module. There exists a quasi-isomorphism $P \xrightarrow{\simeq} X$ from a DG-projective module P; any such morphism is called a *DG-projective resolution* of X. Dually, X has a *DG-injective resolution*, that is, a quasi-isomorphism $X \xrightarrow{\simeq} I$ with I DG-injective. For each X we **fix a DG-projective resolution $\varepsilon^A_X\colon P^A_X \xrightarrow{\simeq} X$ and a DG-injective resolution $\eta^X_A\colon X \xrightarrow{\simeq} I^X_A$.**

Let $\varphi\colon A' \to A$ be a homomorphism of DG algebras, let M and N be A–modules, and let M' and N' be A'–modules. A *φ-contravariant homomorphism* $\nu\colon N \to N'$ is a $\Bbbk$–linear such that $\nu(\varphi(a')n) = (-1)^{|\nu||a'|}a'\nu(n)$ for all $a'\in A'$ and all $n \in N$. A *φ-covariant homomorphism* $\mu\colon M' \to M$ is a $\Bbbk$–linear map such that $\mu(a'm') = (-1)^{|a'||\mu|}\varphi(a')\mu(m')$ for all $a' \in A'$ and all $m' \in M'$. In other words, ν and μ are homomorphisms of A'–modules for the structures induced on N and M through φ.

For φ, μ, ν, as above, and a φ–covariant chain map $\lambda\colon L' \to L$, by 1.2 there exist unique up to homotopy φ–equivariant chain maps $\lambda^\bullet$, $\mu^\bullet$, $\nu_\bullet$, such that the squares

$$
\begin{array}{ccc}
P^{A'}_{L'} & \xrightarrow{\lambda^\bullet} & P^A_L \\
{\simeq}\downarrow & & \downarrow{\simeq} \\
L' & \xrightarrow{\lambda} & L
\end{array}
\qquad
\begin{array}{ccc}
P^{A'}_{M'} & \xrightarrow{\mu^\bullet} & P^A_M \\
{\simeq}\downarrow & & \downarrow{\simeq} \\
M' & \xrightarrow{\mu} & M
\end{array}
\qquad
\begin{array}{ccc}
N & \xrightarrow{\nu} & N' \\
{\simeq}\downarrow & & \downarrow{\simeq} \\
I^N_A & \xrightarrow{\nu_\bullet} & I^{N'}_{A'}
\end{array}
$$

commute up to homotopy. They induce a φ-contravariant chain map

$$\mathrm{Hom}_\varphi(\mu^\bullet,\nu_\bullet)\colon\ \mathrm{Hom}_A(P^A_M, I^N_A) \to \mathrm{Hom}_{A'}(P^{A'}_{M'}, I^{N'}_{A'})\,,\qquad \gamma \mapsto (-1)^{|\mu|(|\gamma|+|\nu|)}\nu_\bullet\gamma\mu^\bullet$$

and a φ–covariant chain map

$$\lambda^\bullet\otimes_\varphi\mu^\bullet\colon P^{A'}_{L'}\otimes_{A'}P^{A'}_{M'} \to P^A_L\otimes_A P^A_M\,,\qquad p'\otimes q' \mapsto (-1)^{|\mu||p'|}\lambda^\bullet(p')\otimes\mu^\bullet(q')$$

both of which are unique up to homotopy.

1.3. **Derived functors.** With the resolutions chosen in 1.2, set $\mathrm{Ext}^*_A(M,N) = \mathrm{H}\,\mathrm{Hom}_A(P^A_M, I^N_A)$ and $\mathrm{Tor}^A_*(L,M) = \mathrm{H}(P^A_L\otimes_A P^A_M)$. The morphisms from 1.2 uniquely define homomorphisms of graded $\Bbbk$–modules

$$\mathrm{Ext}^*_\varphi(\mu,\nu)\colon\ \mathrm{Ext}^*_A(M,N) \to \mathrm{Ext}^*_{A'}(M',N')$$

and

$$\mathrm{Tor}^\varphi_*(\lambda,\mu)\colon\ \mathrm{Tor}^{A'}_*(L',M') \to \mathrm{Tor}^A_*(L,M)$$

which make Ext^* and Tor_* functors of three arguments. If φ, λ, μ, ν, are quasi-isomorphisms, then $\mathrm{Ext}^*_\varphi(\mu,\nu)$ and $\mathrm{Tor}^\varphi_*(\lambda,\mu)$ are isomorphisms.

When $A = A_0$ is a ring, **the construction yields the usual hyper(co)homology of complexes**. By further specialization to modules $M = M_0$ and $N = N_0$, it **produces the classical derived functors of** Hom_A **and** $\otimes_A$.

Let $\gamma \in \operatorname{Ext}^*_A(M,N)$ and $\delta \in \operatorname{Ext}^*_A(L,M)$ be represented by chain maps $\widetilde{\gamma}\colon P^A_M \to I^N_A$ and $\widetilde{\delta}\colon P^A_L \to I^M_A$. By 1.2, choose unique up to homotopy chain maps $\gamma^\bullet\colon P^A_M \to P^A_N$ and $\delta^\bullet\colon P^A_L \to P^A_M$ such that $\widetilde{\gamma} = \eta^N_A \circ \varepsilon^A_N \circ \gamma^\bullet$ and $\widetilde{\delta} = \eta^M_A \circ \varepsilon^A_M \circ \delta^\bullet$.

The *composition product* is the degree 0 homomorphism of graded $\Bbbk$–modules

$$\begin{gathered}\operatorname{Ext}^*_A(M,N) \otimes \operatorname{Ext}^*_A(L,M) \to \operatorname{Ext}^*_A(L,N)\,,\\ \gamma \otimes \delta \mapsto \gamma \cdot \delta = [\widetilde{\gamma} \circ \delta^\bullet]\,.\end{gathered}$$

If t is the homology class of a cycle $\sum_k e_k \otimes_A f_k \in P^A_L \otimes_A P^A_M$, then

$$\begin{gathered}\operatorname{Ext}^*_A(M,N) \otimes \operatorname{Tor}^A_*(L,M) \to \operatorname{Tor}^A_*(L,N)\,,\\ \gamma \otimes t \mapsto \gamma \cdot t = \Big[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes_A \gamma^\bullet(f_k)\Big]\end{gathered}$$

defines a degree 0 homomorphism of graded $\Bbbk$–modules.

Similarly, if u is the homology class of $\sum_k e_k \otimes_A g_k \in P^A_L \otimes_A P^A_N$, then

$$\begin{gathered}\operatorname{Ext}^*_A(L,M) \otimes \operatorname{Tor}^A_*(L,N) \to \operatorname{Tor}^A_*(M,N)\,,\\ \delta \otimes u \mapsto \delta \cdot u = \Big[\sum_k \delta^\bullet(e_k) \otimes_A g_k\Big]\end{gathered}$$

is a degree 0 homomorphism of graded $\Bbbk$–modules.

These pairings are associative, and so define a structure of graded $\Bbbk$–algebra on $\operatorname{Ext}^*_A(M,M)$ with unit given by the class of the map $\eta^M_A \circ \varepsilon^A_M\colon P^A_M \to I^M_A$; they make $\operatorname{Ext}^*_A(M,N)$ into a graded left $\operatorname{Ext}^*_A(N,N)$–, right $\operatorname{Ext}^*_A(M,M)$–bimodule, while $\operatorname{Tor}^A_*(L,M)$ becomes a graded left $\operatorname{Ext}^*_A(L,L)$–, left $\operatorname{Ext}^*_A(M,M)$–bimodule.

2. Construction

We start by describing the setup for much of the discussion in this paper.

2.1. **Koszul regularity.** Let $\boldsymbol{x} = \{x_1, \dots, x_c\}$ be a set of elements in Q, $\mathfrak{a} = (\boldsymbol{x})$ the ideal it generates, and $R = Q/\mathfrak{a}$ the quotient ring. We denote K the *Koszul complex* $K(\boldsymbol{x}, Q)$; thus, $K^\natural$ is the exterior algebra on a free Q–module with basis $\xi_1, \dots, \xi_c$ with $|\xi_j| = 1$, and $\partial(\xi_j) = x_j$ for $1 \le j \le c$. We denote $\mathfrak{A}$ the kernel of the canonical augmentation $\kappa\colon K \to \operatorname{H}_0(K) = R$.

We assume that $\boldsymbol{x}$ is *Koszul-regular* that is, that $\operatorname{H}_i(K) = 0$ for $i \neq 0$; equivalently, κ is a quasi-isomorphism. The *cotangent module* $C = \mathfrak{a}/\mathfrak{a}^2$ is then free over R on $\overline{\boldsymbol{x}}$, where $\overline{x}_j = x_j + \mathfrak{a}^2$, and for each R–module M and each $i \in \mathbb{Z}$ there are isomorphisms

$$\operatorname{Ext}^i_Q(R,M) \cong \operatorname{H}_{-i} \operatorname{Hom}_Q(K,M) = \operatorname{Hom}_Q(K_i,M) \cong \bigwedge^i \operatorname{Hom}_R(C,R) \otimes_R M$$

and

$$\operatorname{Tor}^Q_i(R,M) \cong \operatorname{H}_i(K \otimes_Q M) = K_i \otimes_Q M \cong \bigwedge^i C \otimes_R M\,.$$

The canonical isomorphisms $\kappa^* = \operatorname{Ext}^*_\kappa(M,N)$ and $\kappa_* = \operatorname{Tor}^\kappa_*(L,M)$ are computed as follows. For each R–module X factor its DG resolutions over K as

$$P^K_X \xrightarrow{(\mathrm{id}_X)^\bullet} P^K_X/\mathfrak{A}P^K_X \xrightarrow{(\varepsilon^K_X)^\bullet} X \xrightarrow{(\eta^X_K)_\bullet} (0:\mathfrak{A})_{I^X_K} \xrightarrow{(\mathrm{id}_X)_\bullet} I^X_K \,.$$

Note that the intermediate complexes of R–modules are (respectively) DG-projective and DG-injective and the induced maps $(\varepsilon^K_X)^\bullet$ and $(\eta^X_K)_\bullet$ are quasi-isomorphisms, then choose $\varepsilon^R_X = (\varepsilon^K_X)^\bullet$ and $\eta^X_R = (\eta^X_K)_\bullet$ as the DG resolutions of X over R.

2.2. **Fundamental sequences.** We consider for each j the DG subalgebra $K^{[j]} \subset K$ generated over $K_0 = Q$ by $\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_c$, and the multiplication map

$$\beta_j \colon K \otimes_{K^{[j]}} K \to K \qquad \text{with} \qquad \beta_j(b\otimes c) = bc\,.$$

Proposition. *Let L, M, N, be K–modules.*

(a) *The canonical inclusion $\beta_j|^M_N$ and the degree one map $\alpha_j|^M_N$ defined by*

$$\gamma \longmapsto \big(m \mapsto (-1)^{|\gamma|}\gamma(\xi_j m) - \xi_j\gamma(m)\big)$$

yield an exact sequence of chain maps

$$0 \to \operatorname{Hom}_K(M,N) \xrightarrow{\beta_j|^M_N} \operatorname{Hom}_{K^{[j]}}(M,N) \xrightarrow{\alpha_j|^M_N} \operatorname{Hom}_K(M,N)$$

in which $\alpha_j|^M_N$ is surjective if $M^\natural$ is projective or $N^\natural$ is injective over $K^\natural$.

(b) *The canonical projection $\beta_j|_{LM}$ and the degree one map $\alpha_j|_{LM}$ defined by*

$$l\otimes_K m \mapsto (-1)^{|l|} l \otimes_{K^{[j]}} (\xi_j m) - (\xi_j l)\otimes_{K^{[j]}} m$$

yield an exact sequence of chain maps

$$L\otimes_K M \xrightarrow{\alpha_j|_{LM}} L\otimes_{K^{[j]}} M \xrightarrow{\beta_j|_{LM}} L\otimes_K M \to 0$$

in which $\alpha_j|_{LM}$ is injective if $L^\natural$ or $M^\natural$ is flat over $K^\natural$.

Proof. To simplify notation, we set $A = K^{[j]}$ and $\xi = \xi_j$.

A direct computation yields equalities

$$\operatorname{Ker}\beta_j = (K\otimes_A K)(1\otimes_A \xi - \xi\otimes_A 1) = (0 :_{K\otimes_A K} (1\otimes_A\xi - \xi\otimes_A 1))\,.$$

Thus, left multiplication by $1\otimes_A \xi - \xi\otimes_A 1$ on $K\otimes_A K$ defines a degree 1 homomorphism of graded $K\otimes_A K$–modules

$$\alpha_j \colon K \to K\otimes_A K \qquad \text{given by} \qquad a \mapsto 1\otimes_A(\xi a) - \xi\otimes_A a$$

which fits into a *fundamental exact sequence*

$$0 \to K \xrightarrow{\alpha_j} K\otimes_A K \xrightarrow{\beta_j} K \to 0\,.$$

More verifications show that the action $((b\otimes_A c)\gamma)(m) = (-1)^{|c||\gamma|}b\gamma(cm)$ gives $\operatorname{Hom}_A(M,N)$ a structure of $K\otimes_A K$–module, and that the action $(b\otimes_A c)(l\otimes_A m) = (-1)^{|c||l|}bl\otimes_A cm$ produces such a structure for $L\otimes_A M$. Furthermore, if X is a $K\otimes_A K$–module and K acts on $X\otimes_K M$ by $a(x\otimes m) = (ax)\otimes m$, then there is a canonical isomorphism

$$\operatorname{Hom}_{(K\otimes_A K)}(X, \operatorname{Hom}_A(M,N)) \cong \operatorname{Hom}_K(X\otimes_K M, N)$$

given by $\gamma \longmapsto \big((x\otimes_K m) \mapsto \gamma(x)(m)\big)$, and a canonical isomorphism

$$(L\otimes_A M)\otimes_{(K\otimes_A K)} X \cong L\otimes_K (X\otimes_K M)$$

given by $(l \otimes m) \otimes x \mapsto (-1)^{|m||x|} l \otimes (x \otimes m)$. To obtain the exact sequence in (a), apply the functor $\mathrm{Hom}_{(K \otimes_A K)}(-, \mathrm{Hom}_A(M,N))$ to the fundamental exact sequence, and set $\alpha_j|_N^M = \mathrm{Hom}(\alpha_j, \mathrm{Hom}(M,N))$. For the sequence in (b), apply $(L \otimes_A M) \otimes_{(K \otimes_A K)} -$ to the fundamental sequence, and set $\alpha_j|_{LM} = (L \otimes M) \otimes \alpha_j$. Finally, use the isomorphisms above with $X = K$ to identify the outer terms.

The first isomorphism shows that the sequence in (a) can be obtained in two stages. The first is an application of $- \otimes_K M$ to the fundamental exact sequence, with the K–module structure on $K \otimes_A K$ given by $a(b \otimes c) = (-1)^{|a||b|} b \otimes (ac)$. The map $\beta_j{}^\natural$ is then split by $a \mapsto 1 \otimes a$, so we get an exact sequence of K–modules

$$0 \to M \xrightarrow{\alpha_j|_{KM}} K \otimes_A M \xrightarrow{\beta_j|_{KM}} M \to 0\,.$$

Thus, $\alpha_j|_N^M$ is surjective if $N^\natural$ is injective over $K^\natural$ or $M^\natural$ is projective over $K^\natural$.

A similar argument proves the injectivity of $\alpha_j|_{LM}$ if $L^\natural$ or $M^\natural$ is flat over $K^\natural$. □

2.3. **Cohomology operators.** Let $\boldsymbol{x} = \{x_1, \dots, x_c\} \subset Q$ be a Koszul-regular set. Let L, M, N, be complexes of R–modules. We view them as K–modules via κ and introduce shorthand notation for their resolutions over K

$$U = P_L^K \qquad V = P_M^K \qquad Y = I_K^N\,.$$

Then we choose as in 2.1 resolutions over R

$$E = U/\mathfrak{A}U = P_L^R \qquad F = V/\mathfrak{A}V = P_M^R \qquad J = (0:\mathfrak{A})_Y = I_R^N\,,$$

and let $\varepsilon_U \colon U \to E$, $\varepsilon_V \colon V \to F$, and $\eta^Y : J \to Y$ be the canonical maps.

By 2.2 there are exact sequences of K–modules

$$0 \to \mathrm{Hom}_K(V,Y) \xrightarrow{\beta_j|_Y^V} \mathrm{Hom}_{K[j]}(V,Y) \xrightarrow{\alpha_j|_Y^V} \mathrm{Hom}_K(V,Y) \to 0$$

and

$$0 \to U \otimes_K V \xrightarrow{\alpha_j|_{UV}} U \otimes_{K[j]} V \xrightarrow{\beta_j|_{UV}} U \otimes_K V \to 0\,.$$

The connecting homomorphisms $\eth_j$ of the associated long (co)homology exact sequences define operators χ_j by commutativity of the squares

$$\begin{array}{ccc} \mathrm{Ext}_R^i(M,N) & \xrightarrow{\cong} & \mathrm{Ext}_K^i(M,N) \\ \chi_j \downarrow & & \downarrow \eth_j \\ \mathrm{Ext}_R^{i+2}(M,N) & \xrightarrow{\cong} & \mathrm{Ext}_K^{i+2}(M,N) \end{array} \qquad \begin{array}{ccc} \mathrm{Tor}_i^K(L,M) & \xrightarrow{\cong} & \mathrm{Tor}_i^R(L,M) \\ \eth_j \downarrow & & \downarrow \chi_j \\ \mathrm{Tor}_{i-2}^K(L,M) & \xrightarrow{\cong} & \mathrm{Tor}_{i-2}^R(L,M) \end{array}$$

where the isomorphisms are the maps $\mathrm{Ext}_\kappa^*(M,N)$ and $\mathrm{Tor}_*^\kappa(L,M)$ induced by

$$\mathrm{Hom}_R(F,J) \xrightarrow{\simeq} \mathrm{Hom}_K(V,Y) \qquad \text{with} \qquad \gamma \mapsto \eta^Y \circ \gamma \circ \varepsilon_V$$

and

$$U \otimes_K V \xrightarrow{\simeq} E \otimes_R F \qquad \text{with} \qquad u \otimes_K v \mapsto \varepsilon_U(u) \otimes_R \varepsilon_V(v)\,.$$

We call $\boldsymbol{\chi} = (\chi_j)_1^c$ the *family of cohomology operators* defined by $\boldsymbol{x}$.

For another construction of cohomology operators we study

2.4. **DG-projectives.** Let P be a DG-projective K–module.

The projectivity of $K^\natural$–module $P^\natural$, cf. 1.2, implies that of $\overline{P} = P^\natural / K^\natural_+ P^\natural$ over Q, hence the epimorphism $P^\natural \to \overline{P}$ is split by a homomorphism σ of graded Q–modules. The degree zero homomorphism of $K^\natural$–modules $\pi \colon K^\natural \otimes_Q \operatorname{Im}(\sigma) \to P^\natural$ given by $\pi(a \otimes p) = ap$ induces an isomorphism $\pi \otimes_{K^\natural} Q$, so $\operatorname{Coker}(\pi) = 0$. Thus, we have an exact sequence of graded $K^\natural$–modules

$$0 \to \operatorname{Ker} \pi \to K^\natural \otimes_Q \operatorname{Im}(\sigma) \xrightarrow{\pi} P^\natural \to 0$$

which splits because $P^\natural$ is projective. In the exact sequence obtained by applying $- \otimes_{K^\natural} Q$ the map $\pi \otimes_{K^\natural} Q$ is bijective, so $\operatorname{Ker}(\pi) \otimes_{K^\natural} Q = 0$. Due to the nilpotency of the ideal $K^\natural_+ = (\xi_1, \dots, \xi_c) K^\natural$, we have $\operatorname{Ker}(\pi) = 0$, hence π is bijective.

For $\{h_1, \dots, h_i\} = H \subseteq Z = \{1, \dots, c\}$ with $h_1 < \cdots < h_i$, set $\xi_H = \xi_{h_1} \wedge \cdots \wedge \xi_{h_i} \in K$. Each $v \in P$ can be written in the form $v = \sum_{H \subseteq Z} \xi_H \langle v \rangle_H$ with uniquely defined $\langle v \rangle_H \in \operatorname{Im}(\sigma)$, so for each $H \subseteq Z$ we have a Q–linear endomorphism

$$\partial_H \colon \operatorname{Im}(\sigma) \to \operatorname{Im}(\sigma) \qquad \text{with} \qquad \partial_H(p) = (-1)^{|H|} \langle \partial p \rangle_H$$

of degree $(-|H| - 1)$. The equation $\partial^2(p) = 0$ then yields

$$\partial^2_\varnothing(p) = \sum_{j=1}^{c} x_j \partial_j(p)\,; \tag{2.4.1}$$

$$\partial_\varnothing \partial_j(p) \equiv \partial_j \partial_\varnothing(p) \mod \mathfrak{a} \operatorname{Im}(\sigma) \qquad \text{for} \quad j = 1, \dots, c\,. \tag{2.4.2}$$

Note that $D = R \otimes_K P \cong \operatorname{Im}(\sigma)/\mathfrak{a}\operatorname{Im}(\sigma) = P/\mathfrak{A}P$ is a complex of projective R–modules with differential $\partial(1 \otimes p) = 1 \otimes \partial_\varnothing(p)$, that the maps

$$\tau^D_j \colon D \to D \qquad \text{given by} \qquad 1 \otimes p \mapsto 1 \otimes \partial_j(p) \qquad \text{for} \quad j = 1, \dots, c$$

are degree -2 chain endomorphisms of D, and that the family $\boldsymbol{\tau}^D = (\tau^D_j)^c_1$ is uniquely determined by $\boldsymbol{x}$. It allows for chain level computations of the operators.

Proposition. *Let L, M, N, be complexes of R–modules, and let $\varepsilon^R_L \colon E \to L$, $\varepsilon^R_M \colon F \to M$, $\eta^M_R \colon M \to I$, be their DG resolutions over R described in* 2.3.

If $\tau^E_1, \dots, \tau^E_c \colon E \to E$ and $\tau^F_1, \dots, \tau^F_c \colon F \to F$ are families of chain endomorphisms described above, then for $j = 1, \dots, c$ there are equalities

$$\begin{aligned}
\chi_j &= \operatorname{H} \operatorname{Hom}_R(\tau^F_j, J) \colon \operatorname{Ext}^*_R(M, N) \to \operatorname{Ext}^*_R(M, N)\,;\\
\chi_j &= \quad \operatorname{H}(\tau^E_j \otimes_R F) \colon \operatorname{Tor}^R_*(L, M) \to \operatorname{Tor}^R_*(L, M)\,;\\
\chi_j &= -\operatorname{H}(E \otimes_R \tau^F_j) \colon \operatorname{Tor}^R_*(L, M) \to \operatorname{Tor}^R_*(L, M)\,.
\end{aligned}$$

Proof. Along with the chosen in 2.3 DG-projective resolutions V and F of M over K and R, we consider the complex of R–modules $F^{[j]} = R \otimes_{K^{[j]}} V$. Identifying $F^{[j]\natural}$ with $F^\natural \oplus \xi_j F^\natural$, we see that its differential is given by

$$\partial(f' + \xi_j f'') = \partial(f') - \xi_j \big(\tau^F_j(f') + \partial(f'')\big) \quad \text{for } f', f'' \in F\,.$$

Due to 2.2 we have a commutative diagram of chain maps with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{\alpha_j|_{KV}} & K \otimes_{K^{[j]}} V & \xrightarrow{\beta_j|_{KV}} & V & \longrightarrow & 0\\
 & & \simeq\big\downarrow & & \big\downarrow\simeq & & \big\downarrow\simeq & & \\
0 & \longrightarrow & F & \xrightarrow{\ddot{\alpha}_j|_{RF}} & F^{[j]} & \xrightarrow{\ddot{\beta}_j|_{RF}} & F & \longrightarrow & 0
\end{array} \tag{2.4.3}$$

where the middle vertical arrow is $\kappa\otimes_{K^{[j]}}V$ and the maps in the bottom row are $\ddot\alpha_j|_{RF}(f)=\xi_j f$ and $\ddot\beta_j|_{RF}(f'+\xi_j f'')=f'$.

Apply $\operatorname{Hom}_K(-,Y)$ to (2.4.3). As Y is DG-injective over K, this functor is exact. Furthermore, on complexes of R–modules it reduces to $\operatorname{Hom}_R(-,J)$, so we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_K(V,Y) & \xrightarrow{\beta_j|^V_Y} & \operatorname{Hom}_{K^{[j]}}(V,Y) & \xrightarrow{\alpha_j|^V_Y} & \operatorname{Hom}_K(V,Y) & \longrightarrow & 0\\
 & & \simeq\big\uparrow & & \big\uparrow\simeq & & \big\uparrow\simeq & & \\
0 & \longrightarrow & \operatorname{Hom}_R(F,J) & \xrightarrow{\ddot\beta_j|^F_J} & \operatorname{Hom}_R(F^{[j]},J) & \xrightarrow{\ddot\alpha_j|^F_J} & \operatorname{Hom}_R(F,J) & \longrightarrow & 0
\end{array}
$$

and chain maps $\ddot\beta_j|^F_J(\widetilde\gamma)(f'+\xi_j f'')=\widetilde\gamma(f')$ and $\ddot\alpha_j|^F_J(\widetilde\gamma)(f)=(-1)^{|\gamma|}\widetilde\gamma(\xi_j f)$. Its commutativity shows that the operator χ_j on $\operatorname{Ext}^*_R(M,N)$ can be computed by the connecting homomorphism $\ddot{\eth}_j$ of the lower row. A direct computation shows that if γ is the class of a chain map $\widetilde\gamma\colon F\to J$, then $\ddot{\eth}_j(\gamma)=[\widetilde\gamma\circ\tau^F_j]$. In other words, $\chi_j(\gamma)=\operatorname{H}\operatorname{Hom}_R(\tau^F_j,J)(\gamma)$.

Similar considerations applied to the DG-projective resolutions U and E of L over K and R produce a commutative diagram of chain maps with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{\dot\alpha_j|_{ER}} & E^{[j]} & \xrightarrow{\dot\beta_j|_{ER}} & E & \longrightarrow & 0\\
 & & \simeq\big\uparrow & & \big\uparrow\simeq & & \big\uparrow\simeq & & \\
0 & \longrightarrow & U & \xrightarrow{\alpha_j|_{UK}} & U\otimes_{K^{[j]}}K & \xrightarrow{\beta_j|_{UK}} & U & \longrightarrow & 0
\end{array}
\tag{2.4.4}
$$

with $E^{[j]}=E\otimes_{K^{[j]}}R$. In it the middle quasi-isomorphism is $U\otimes_{K^{[j]}}\kappa$, and the maps in the top row are $\dot\alpha_j|_{ER}(e)=-\xi_j e$ and $\dot\beta_j|_{ER}(e'+\xi_j e'')=e'$.

Applying $(-\otimes_K V)$ to (2.4.4) and $(U\otimes_K -)$ to (2.4.2), we arrive at diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E\otimes_R F & \xrightarrow{\dot\alpha_j|_{EF}} & E^{[j]}\otimes_R F & \xrightarrow{\dot\beta_j|_{EF}} & E\otimes_R F & \longrightarrow & 0\\
 & & \simeq\big\uparrow & & \big\uparrow\simeq & & \big\uparrow\simeq & & \\
0 & \longrightarrow & U\otimes_R V & \xrightarrow{\alpha_j|_{UV}} & U\otimes_{K^{[j]}}V & \xrightarrow{\beta_j|_{UV}} & U\otimes_R V & \longrightarrow & 0\\
 & & \simeq\big\downarrow & & \big\downarrow\simeq & & \big\downarrow\simeq & & \\
0 & \longrightarrow & E\otimes_R F & \xrightarrow{\ddot\alpha_j|_{EF}} & E\otimes_R F^{[j]} & \xrightarrow{\ddot\beta_j|_{EF}} & E\otimes_R F & \longrightarrow & 0
\end{array}
$$

which is commutative with exact rows. By the bottom portion, χ_j on $\operatorname{Tor}^R_*(L,M)$ is the connecting homomorphism $\ddot{\eth}_j$ of a short exact sequence of complexes in which $\ddot\alpha_j|_{EF}(e\otimes f)=(-1)^{|e|}e\otimes(\xi_j f)$ and $\ddot\beta_j|_{EF}(e\otimes(f'+\xi_j f''))=e\otimes f'$. By an easy calculation $\ddot{\eth}_j\big(\big[\sum_k e_k\otimes f_k\big]\big)=-\big[\sum_k e_k\otimes\tau^F_j(f_k)\big]$, hence $\chi_j=-\operatorname{H}(F\otimes_R\tau^F_j)$.

The top part shows that the action of χ_j on $\operatorname{Tor}^R_*(L,M)$ is also given by the connecting homomorphism $\dot{\eth}_j$ of a sequence with $\dot\alpha_j|_{EF}(e\otimes f)=-(\xi_j e)\otimes f$, and $\dot\beta_j|_{EF}((e'+\xi_j e'')\otimes f)=e'\otimes f$. This time, $\dot{\eth}_j\big(\big[\sum_k e_k\otimes f_k\big]\big)=\big[\sum_k\tau^E_j(e_k)\otimes f_k\big]$. □

Corollary. *If $\eta^M_R\colon M\to I$ is a DG-injective resolution over R and $1_M=[\eta^M_R\circ\varepsilon^R_M]\in\operatorname{Ext}^0_R(M,M)$, then $\chi_j(1_M)=[\eta^M_R\circ\varepsilon^R_M\circ\tau^F_j]\in\operatorname{Ext}^2_R(M,M)$ for $j=1,\dots,c$.* □

An alternative approach to operators in cohomolgy is obtained by using

2.5. **DG-injectives.** In addition to the assumptions and notation of 2.1 and 2.3, let $\{\xi_H^*\}_{H\subseteq Z}$ be the Q–basis of $\operatorname{Hom}_Q(K^\natural, Q)$ dual to the basis $\{\xi_H\}_{H\subseteq Z}$ of $K^\natural$. The action of $K^\natural$ on $\operatorname{Hom}_Q(K^\natural, Q)$ is described by $\xi_j\xi_H^* = 0$ if $j \notin H$, and $\xi_j\xi_H^* = (-1)^{r-1}\xi_{H\smallsetminus j}^*$ if $j = h_r$, where $H = \{h_1, \dots, h_i\}$ and $h_1 < \dots < h_i$.

Let I be a DG-injective K–module.

The graded module $\underline{I} = (0 : K_+^\natural)_{I^\natural}$ is then injective over Q, due to the injectivity of $I^\natural$ over $K^\natural$, cf. 1.2. Let ρ be a Q–linear splitting of $\underline{I} \subseteq I^\natural$, and set $\iota(y)(a) = (-1)^{|y||a|}\rho(ay)$ for $y \in I^\natural$ and $a \in K^\natural$. Therefore, $\iota\colon I^\natural \to \operatorname{Hom}_Q(K^\natural, \underline{I})$ is a degree 0 homomorphism of graded $K^\natural$–modules. As $\operatorname{Hom}_{K^\natural}(Q,\iota)$ is bijective, we see that $\operatorname{Ker}(\iota) = 0$, so ι is injective and yields an exact sequence of graded $K^\natural$–modules

$$0 \to I^\natural \xrightarrow{\iota} \operatorname{Hom}_Q(K^\natural, \underline{I}) \to \operatorname{Coker}\iota \to 0\,.$$

The sequence splits, due to the injectivity of $I^\natural$. Applying to it $\operatorname{Hom}_{K^\natural}(Q,-)$ we get an exact sequence whose first map is bijective, so $\operatorname{Hom}_{K^\natural}(Q, \operatorname{Coker}\iota) = 0$. We remark that if X is a $K^\natural$–module such that $\operatorname{Hom}_{K^\natural}(Q, X) = 0$, then the nilpotency of the graded ideal $K_+^\natural$ implies that $X = 0$. Thus, $\operatorname{Coker}\iota = 0$ and ι is bijective.

The map $\varkappa\colon K^\natural \otimes_Q \underline{I} \to \operatorname{Hom}_Q(K^\natural, Q) \otimes_Q \underline{I}$ with $a \otimes w \mapsto (-1)^{c|a|}a\xi_Z^* \otimes w$ is a bijective degree $(-c)$ homomorphism of graded $K^\natural$–modules. Thus, $y \in I$ has a unique expression of the form $y = \sum_{H\subseteq Z}\xi_H\langle y\rangle_H$ with $\langle y\rangle_H \in W = \iota^{-1}\varkappa(1 \otimes_Q \underline{I})$. In particular, for each $y \in \underline{I}$ there is a unique $w \in W$ such that $y = \xi_Z w$. For $H \subseteq Z$ we define a Q–linear degree $(-|H| - 1)$ endomorphism

$$\partial_H : \underline{I} \to \underline{I} \qquad \text{with} \qquad \partial_H(y) = (-1)^{c-|H|}\xi_Z\langle \partial w\rangle_H\,.$$

By a direct computation, the equation $\partial^2(w) = 0$ now yields a congruence

$$\langle\partial\langle\partial w\rangle_\varnothing\rangle_\varnothing \equiv -\sum_{j=1}^{c} x_j\langle\partial w\rangle_j + \sum_{j=1}^{c}\xi_j\big(\langle\partial\langle\partial w\rangle_j\rangle_\varnothing - \langle\partial\langle\partial w\rangle_\varnothing\rangle_j\big)$$
$$\mod \left(\sum_{\{h,j\}\subseteq Z}\xi_j\mathfrak{a}\langle\partial w\rangle_{\{h,j\}} + \sum_{|H|\geqslant 2}\xi_H W\right).$$

Multiplication on the left with ξ_Z gives $\xi_Z\langle\partial\langle\partial w\rangle_\varnothing\rangle_\varnothing = -\xi_Z\sum_{j=1}^{c} x_j\langle\partial w\rangle_j$. Thus,

$$\partial_\varnothing^2(y) = (-1)^c\sum_{j=1}^{c} x_j\partial_j(y)$$

and the congruence above simplifies to

$$\sum_{j=1}^{c}\xi_j\big(\langle\partial\langle\partial w\rangle_\varnothing\rangle_j - \langle\partial\langle\partial w\rangle_j\rangle_\varnothing\big) \equiv 0 \mod \left(\sum_{\{h,j\}\subseteq Z}\xi_j\mathfrak{a}\langle\partial w\rangle_{\{h,j\}} + \sum_{|H|\geqslant 2}\xi_H W\right).$$

Multiplying the new congruence on the left with $\xi_{Z\smallsetminus j}$ for $j = 1, \dots, c$, we get

$$\partial_j\partial_\varnothing(y) \equiv \partial_\varnothing\partial_j(y) \mod \left(\sum_{h=1}^{c}\mathfrak{a}\langle\partial w\rangle_{\{h,j\}}\right).$$

Remark that y is in $B = (0 : \mathfrak{a})_{\underline{I}} = (0 : \mathfrak{A})_I$ if and only if w is in $(0 : \mathfrak{a})_W$, and that the restriction of $\partial_\varnothing$ to B is equal to the differential ∂ of B. Thus, setting $\upsilon_j = (-1)^c\partial_j$ and denoting υ_j^B the restriction of υ_j to B, we obtain a family $\boldsymbol{\upsilon}^B = (\upsilon_j^B)_1^c$ of chain endomorphisms of degree -2 of the complex of R–modules B.

Proposition. *Let M and N be complexes of R–modules, let $F = V/\mathfrak{A}V$ and $J = (0:\mathfrak{A})_Y$ be their DG resolutions as in 2.3, and let $\upsilon_1^J,\dots,\upsilon_c^J\colon J \to J$ be the family of chain endomorphisms described above. For $j=1,\dots,c$ there are equalities*

$$\chi_j = \mathrm{H}\,\mathrm{Hom}_R(F,\upsilon_j^J)\colon\ \mathrm{Ext}_R^*(M,N) \to \mathrm{Ext}_R^*(M,N)\,.$$

Proof. Consider the complex of R–modules $J^{[j]} = \mathrm{Hom}_{K^{[j]}}(R,Y)$, and identify $J^{[j]\natural}$ with $\xi_Z(0:\mathfrak{a})_W^\natural \oplus \xi_{Z\smallsetminus j}(0:\mathfrak{a})_W^\natural$. The differential of $J^{[j]}$ is then given by

$$\partial(\xi_Z w' + \xi_{Z\smallsetminus j}w'') = \partial_J(\xi_Z w') + (-1)^j \upsilon_j(\xi_Z w'') + (-1)^{c-1}\xi_{Z\smallsetminus j}\langle \partial w''\rangle_{\varnothing}$$

for $w', w'' \in (0:\mathfrak{a})_W^\natural$. Thus, from 2.2 we get a commutative diagram of chain maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \xrightarrow{\dot\beta_j|_J^R} & J^{[j]} & \xrightarrow{\dot\alpha_j|_J^R} & J & \longrightarrow & 0\\
 & & \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq & & \\
0 & \longrightarrow & Y & \xrightarrow{\beta_j|_Y^K} & \mathrm{Hom}_{K^{[j]}}(K,Y) & \xrightarrow{\alpha_j|_Y^K} & Y & \longrightarrow & 0
\end{array}$$

in which the rows are exact, the middle vertical arrow is induced by $\mathrm{Hom}_{K^{[j]}}(\kappa, Y)$ and is a quasi-isomorphism because the external two arrows are, and where

$$\dot\beta_j|_J^R(\xi_Z w) = \xi_Z w \qquad \text{and} \qquad \dot\alpha_j|_J^R(\xi_Z w' + \xi_{Z\smallsetminus j}w'') = (-1)^j \xi_Z w''\,.$$

The rest of the argument is similar to that for Proposition 2.4. □

3. Properties

We derive the main formal properties of the cohomology operators.

3.1. **Naturality.** Given a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}
Q' & \xrightarrow{\psi} & Q\\
\rho'\downarrow & & \downarrow\rho\\
R' & \xrightarrow{\varphi} & R
\end{array}$$

with $\mathfrak{a}' = \mathrm{Ker}(\rho')$ generated by a Koszul-regular set $\boldsymbol{x}' = x_1',\dots,x_{c'}' \subset Q'$, let $\boldsymbol{\chi}' = \{\chi_1',\dots,\chi_{c'}'\}$ be the family of cohomology operators defined by $\boldsymbol{x}'$, and let

$$\psi(x_i') = \sum_{j=1}^{c} q_{ij}x_j \qquad \text{with} \qquad q_{ij}\in Q \qquad \text{for} \quad 1 \le i \le c'$$

be equalities resulting from the inclusion $\psi(\boldsymbol{x}') \subset \mathrm{Ker}(\rho)$.

We can now describe the functoriality of the cohomology operators.

Theorem. *Let $\mu\colon M' \to M$ be φ-covariant chain map.*

If $\lambda\colon L' \to L$ is a φ-covariant chain map, then

$$\chi_j \circ \mathrm{Tor}_*^{\varphi}(\lambda,\mu) = \sum_{i=1}^{c'} q_{ij}\,\mathrm{Tor}_*^{\varphi}(\lambda,\mu)\circ\chi_i' \qquad \textit{for} \quad j=1,\dots,c\,.$$

If $\nu\colon N \to N'$ is a φ-contravariant chain map and $q_{ij} = \psi(q_{ij}')$, then

$$\sum_{i=1}^{c'} q_{ij}'\chi_i' \circ \mathrm{Ext}_{\varphi}^*(\mu,\nu) = \mathrm{Ext}_{\varphi}^*(\mu,\nu)\circ\chi_j \qquad \textit{for} \quad j=1,\dots,c\,.$$

Proof. This is immediate from 1.2, Proposition 2.4, and the following lemma. □

Lemma. *Let K' be the Koszul complex $K(x',Q')$, and let $\Psi\colon K'\to K$ be the homomorphism of DG algebras defined by $\Psi(\xi_i')=\sum_{j=1}^{c} q_{ij}\xi_j$ for $1\le i\le c'$. Let P' be a DG-projective module over K', and let $\boldsymbol{\tau}'=(\tau_i')_1^{c'}$ be the family of chain endomorphisms of the complex of R'–modules $D'=P'/\mathfrak{A}'P'$ defined by $\boldsymbol{x}'$. Given a Ψ-covariant chain map $\gamma\colon P'\to P$, let $1\otimes\gamma\colon D'\to D$ be the induced covariant chain map of complexes over the ring homomorphism $\varphi\colon R'\to R$.*

The chain maps $\tau_j^D\circ(1\otimes\gamma)$ and $\sum_{i=1}^{c'} q_{ij}(1\otimes\gamma)\circ\tau_i^{D'}$ are homotopic.

Remark. In view of Proposition 4.2 the Lemma is equivalent to [7, Proposition 1.7].

Proof. In the notation of 2.4, for each $H\subseteq Z=\{1,\dots,c\}$ consider the degree $(|\gamma|-|H|)$ homomorphisms of graded Q'–modules

$$\gamma_H\colon \operatorname{Im}(\sigma')\to\operatorname{Im}(\sigma)\qquad\text{with}\qquad \gamma_H(y')=(-1)^{|H|}\langle\gamma(y')\rangle_H\,.$$

By definition, we have

$$\begin{aligned}\partial\gamma(y')&\equiv\partial_\varnothing\gamma_\varnothing(y')-\sum_{j=1}^{c}x_j\gamma_j(y')\\&+\sum_{j=1}^{c}\xi_j\Big(\partial_\varnothing\gamma_j(y')-\partial_j\gamma_\varnothing(y')+\sum_{\{h,j\}\subseteq Z}\pm x_h\gamma_{\{h,j\}}(y')\Big)\quad\operatorname{mod}\Big(\sum_{|H|\geqslant 2}\xi_HP\Big).\end{aligned}$$

Due to the Ψ-covariance of γ, we get

$$\begin{aligned}\gamma\partial'(y')&\equiv\gamma_\varnothing\partial'_\varnothing(y')-\sum_{j=1}^{c}\xi_j\gamma_j\partial'_\varnothing(y')-(-1)^{|\gamma|}\sum_{i=1}^{c'}\Psi(\xi_i')\,\gamma_\varnothing\partial_i'(y')\\&=\gamma_\varnothing\partial'_\varnothing(y')+\sum_{j=1}^{c}\xi_j\Big((-1)^{|\gamma|-1}\sum_{i=1}^{c'}q_{ij}\gamma_\varnothing\partial_i'(y')-\gamma_j\partial'_\varnothing(y')\Big)\quad\operatorname{mod}\Big(\sum_{|H|\geqslant 2}\xi_HP\Big).\end{aligned}$$

Noting that $\partial\gamma=(-1)^{|\gamma|}\gamma\partial'$, and that $\gamma_\varnothing$ induces the morphism $1\otimes\gamma\colon D'\to D$, we conclude by comparing the coefficients of ξ_j that

$$\tau_j^D\circ(1\otimes\gamma)-\sum_{i=1}^{c'}q_{ij}(1\otimes\gamma)\circ\tau_i^{D'}=\partial\gamma_j-(-1)^{|\gamma_j|}\gamma_j\partial'\quad\text{for } j=1,\dots,c\,.$$

Thus, γ_j is a homotopy from $\tau_j^D\circ(1\otimes\gamma)$ to $\sum_{i=1}^{c'}q_{ij}(1\otimes\gamma)\circ\tau_i^{D'}$. □

3.2. **Centrality.** Cohomology operators (anti)commute with products:

Theorem. *For $j=1,\dots,c$ there are equalities*

$$\begin{aligned}\chi_j(\gamma)\cdot\delta&=\ \ \chi_j(\gamma\cdot\delta)&&=\ \ \gamma\cdot\chi_j(\delta)&&\in\operatorname{Ext}_R^*(L,N)\\\chi_j(\gamma)\cdot t&=-\chi_j(\gamma\cdot t)&&=-\gamma\cdot\chi_j(t)&&\in\operatorname{Tor}^R_*(L,N)\\\chi_j(\delta)\cdot u&=\ \ \chi_j(\delta\cdot u)&&=\ \ \delta\cdot\chi_j(u)&&\in\operatorname{Tor}^R_*(M,N)\end{aligned}$$

for all $\gamma\in\operatorname{Ext}_R^*(M,N)$, $\delta\in\operatorname{Ext}_R^*(L,M)$, $t\in\operatorname{Tor}^R_*(L,M)$, $u\in\operatorname{Tor}^R_*(L,N)$.

Proof. In addition to the resolutions F, E, J, described in 2.3, we consider DG resolutions $M\xrightarrow{\simeq}I_R^M=I$ and $G=P_N^R\xrightarrow{\simeq}N$.

Choose chain maps $\widetilde{\gamma}\colon F \to J$ and $\widetilde{\delta}\colon E \to I$ such that $\gamma = [\widetilde{\gamma}]$ and $\delta = [\widetilde{\delta}]$, cf. 1.2. Extend $\widetilde{\gamma}$ to $\widetilde{\gamma}_{\bullet} \in \operatorname{Hom}_R(I, J)$ and lift $\widetilde{\delta}$ to $\widetilde{\delta}^{\bullet} \in \operatorname{Hom}_R(E, F)$. Lemma 3.1 applied to id_K shows that $\tau_j^F \circ \widetilde{\delta}^{\bullet}$ and $\widetilde{\delta}^{\bullet} \circ \tau_j^E$ are homotopic, so Proposition 2.4 gives

$$\chi_j(\gamma) \cdot \delta = [\widetilde{\gamma} \circ \tau_j^F] \cdot \delta = [\widetilde{\gamma} \circ (\tau_j^F \circ \widetilde{\delta}^{\bullet})] = [\widetilde{\gamma} \circ (\widetilde{\delta}^{\bullet} \circ \tau_j^E)] = [(\widetilde{\gamma} \circ \widetilde{\delta}^{\bullet}) \circ \tau_j^E] = \chi_j(\gamma \cdot \delta)\,.$$

On the other hand, by [6, §7, no. 2] and Proposition 2.4 we have

$$\chi_j(\gamma \cdot \delta) = \chi_j([\widetilde{\gamma} \circ \widetilde{\delta}^{\bullet}]) = [(\widetilde{\gamma} \circ \widetilde{\delta}^{\bullet}) \circ \tau_j^E] = [(\widetilde{\gamma}_{\bullet} \circ \widetilde{\delta}) \circ \tau_j^E] = \gamma \cdot [\widetilde{\delta}^{\bullet} \circ \tau_j^E] = \gamma \cdot \chi_j(\delta)\,.$$

With $\sum_k e_k \otimes f_k$ representing t and $\widetilde{\gamma}^{\bullet}\colon F \to G$ lifting $\widetilde{\gamma}$, Proposition 2.4 yields

$$\chi_j(\gamma) \cdot t = [\gamma \circ \tau_j^F] \cdot \Big[\sum_k e_k \otimes f_k \Big] = \Big[\sum_k (-1)^{(|\gamma|-2)|e_k|} e_k \otimes \widetilde{\gamma}^{\bullet}\big(\tau_j^F(f_k)\big) \Big]\,;$$

$$\chi_j(\gamma \cdot t) = \chi_j\bigg(\Big[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \widetilde{\gamma}^{\bullet}(f_k) \Big] \bigg) = -\Big[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \tau_j^G\big(\widetilde{\gamma}^{\bullet}(f_k)\big) \Big]\,;$$

$$\gamma \cdot \chi_j(t) = \gamma \cdot \Big[-\sum_k e_k \otimes \big(\tau_j^F(f_k)\big) \Big] = -\Big[\sum_k (-1)^{|\gamma||e_k|} e_k \otimes \widetilde{\gamma}^{\bullet}\big(\tau_j^F(e_k)\big) \Big]\,.$$

As $\tau_j^G \circ \widetilde{\gamma}^{\bullet}$ and $\widetilde{\gamma}^{\bullet} \circ \tau_j^F$ are homotopic by Lemma 3.1, we get the second statement.

Finally, if $\sum_k e_k \otimes g_k$ represents u, then by Proposition 2.4 we have

$$\chi_j(\delta) \cdot u = [\widetilde{\delta} \circ \tau_j^E] \cdot \Big[\sum_k e_k \otimes g_k \Big] = \Big[\sum_k \big(\widetilde{\delta}^{\bullet} \circ \tau_j^E(e_k)\big) \otimes g_k \Big]\,;$$

$$\chi_j(\delta \cdot u) = \chi_j\bigg(\Big[\sum_k \big(\widetilde{\delta}^{\bullet}(e_k)\big) \otimes g_k \Big] \bigg) = \Big[\sum_k \big(\tau_j^F \circ \widetilde{\delta}^{\bullet}(e_k)\big) \otimes g_k \Big]\,;$$

$$\delta \cdot \chi_j(u) = \delta \cdot \Big[\sum_k \big(\tau_j^E(e_k)\big) \otimes g_k \Big] = \Big[\sum_k \big(\widetilde{\delta}^{\bullet} \circ \tau_j^E(e_k)\big) \otimes g_k \Big]\,.$$

As $\tau_j^F \circ \widetilde{\delta}^{\bullet}$ and $\widetilde{\delta}^{\bullet} \circ \tau_j^E$ are homotopic, we have the last statement. □

3.3. **Characteristic homomorphism.** Set $C = \mathfrak{a}/\mathfrak{a}^2$, and let $x_1^*, \dots, x_c^*$ denote the basis of $C^* = (\mathfrak{a}/\mathfrak{a}^2)^*$ dual to the basis $\overline{\boldsymbol{x}} = \{x_1 + \mathfrak{a}^2, \dots, x_c + \mathfrak{a}^2\}$ of C.

Theorem. *The unique homomorphism of R–modules*

$$C^* \to \operatorname{Ext}_R^2(M, M) \qquad \textit{with} \qquad x_j^* \mapsto \chi_j(1_M) \qquad \textit{for} \quad j = 1, \dots, c$$

is independent of the choice of the Koszul-regular set $\boldsymbol{x}$. Its image lies in the center of $\operatorname{Ext}_R^*(M, M)$, and so it extends canonically to a characteristic homomorphism *of graded R–algebras from the symmetric algebra $\mathcal{S} = \operatorname{Sym}_R(C^*)$:*

$$\zeta_M\colon \mathcal{S} \to \operatorname{Ext}_R^*(M, M)$$

The maps ζ_M and ζ_N induce on $\operatorname{Ext}_R^*(M, N)$ the same $\mathcal{S}$–module structure:

$$\zeta_N(x_j^*) \cdot \gamma = \chi_j(\gamma) = \gamma \cdot \zeta_M(x_j^*) \qquad \textit{for} \quad \gamma \in \operatorname{Ext}_R^*(M, N)\,,$$

The maps ζ_L and ζ_M induce on $\operatorname{Tor}_*^R(L, M)$ opposite $\mathcal{S}$–module structures:

$$\zeta_L(x_j^*) \cdot t = \chi_j(t) = -\zeta_M(x_j^*) \cdot t \qquad \textit{for} \quad t \in \operatorname{Tor}_*^R(L, M)\,.$$

Corollary. *On $\operatorname{Ext}_R^*(M, N)$ and $\operatorname{Tor}_*^R(L, M)$ there are equalities*

$$\chi_i \chi_j = \chi_j \chi_i \qquad \textit{for} \quad i, j = 1, \dots, c\,. \qquad \square$$

Remark. When M is an R–*module* the characteristic homomorphism ζ_M coincides with one constructed by Mehta, cf. 4.4; his definition does not seem to generalize to complexes of R–modules. When $c = 1$, he proves in [9, Proposition 2.3] that the image of ζ_M is central and in [9, Proposition 2.4] that over complete local rings the actions on Tor coincide. The theorem extends (and corrects some signs of) *loc. cit.*

Proof of the Theorem. Theorem 3.2 yields the formulas for the action of $\zeta_M(x_j^*)$, and the first one of them implies that the image of ζ_M is in the center of of $\operatorname{Ext}_R^*(M,M)$.

Along with $\boldsymbol{x}$, consider a Koszul-regular set $\boldsymbol{x}' \subset Q$ such that $(\boldsymbol{x}') = (\boldsymbol{x})$. As both $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{x}}'$ are bases of C, the set $\boldsymbol{x}'$ also consists of c elements, say $x'_1,\dots,x'_c$. Let ${\tau'_1}^F,\dots,{\tau'_c}^F$ be the chain endomorphisms of 2.4 defined by $\boldsymbol{x}'$ on the chosen DG projective resolution F of M over R. Writing $x'_i = \sum_{j=1}^c q_{ij}x_j$, and applying Lemma 3.1 to the identity maps of K and of $P = P_M^K$, we get

$$\tau_j^F = \sum_{i=1}^{c} q_{ij}{\tau'_i}^F \qquad \text{for} \quad j = 1,\dots,c\,.$$

It follows that the degree zero R–linear homomorphism $(\mathfrak{a}/\mathfrak{a}^2)^* \to \operatorname{Hom}_R(F,F)$ with $x_j^* \mapsto \tau_j^F$ for $j = 1,\dots,c$ does not depend on the choice of the Koszul-regular set $\boldsymbol{x}$. In view of the definition of ζ_M and Corollary 2.4 we have

$$\zeta_M(x_j^*) = [\eta_R^M \circ \varepsilon_M^R \circ \tau_j^F] = \chi_j(1_M).$$

This establishes the independence of ζ_M from the choice of $\boldsymbol{x}$. □

4. Comparison

We present the cohomology operators constructed by Gulliksen [8], Eisenbud [7], Mehta [9], Avramov [1], and compare them to the operators χ_j from 2.3.

Throughout this section $\rho\colon Q \to R$ is a surjective homomorphism of commutative rings with $\operatorname{Ker}(\rho) = \mathfrak{a}$ generated by a Koszul-regular set $\boldsymbol{x} = x_1,\dots,x_c$. We consider R–*modules* L, M, N; as a further adjustment, we assume that DG-resolutions U, V, Y of L, M, N over K have been chosen with $U_i = 0$ and $V_i = 0$ for $i < 0$, respectively $Y_i = 0$ for $i > 0$. This can always be done, cf. [3], and has the effect that the chosen in 2.3 complexes $F = U/\mathfrak{A}U$ and $E = V/\mathfrak{A}V$ are classical projective resolutions of the R–modules L and M, while the complex $J = (0\colon \mathfrak{A})_Y$ is a classical injective resolution of the R–module module N. Finally, recall that

$$\operatorname{Ext}_R^*(M,N) = \operatorname{H}\operatorname{Hom}_R(F,J) \qquad \text{and} \qquad \operatorname{Tor}_*^R(L,M) = \operatorname{H}(E\otimes_R F).$$

4.1. Gulliksen's construction [8, p. 176] starts from short exact sequences

$$0 \to E \xrightarrow{\alpha'_j} E^{[j]} \xrightarrow{\beta'_j} E \to 0$$

and

$$0 \to F \xrightarrow{\alpha''_j} F^{[j]} \xrightarrow{\beta''_j} F \to 0$$

where $E^{[j]}$ and $F^{[j]}$ are as in 2.4; the maps in the first sequence are given by $\alpha'_j(e) = (-1)^{|e|}\xi_j e$ and $\beta'_j(e' + \xi_j e'') = e'$; the second sequence is defined similarly.

The preceding sequences yield exact sequences of (anti)chain[1] maps

$$0 \to \operatorname{Hom}_R(F,N) \xrightarrow{\operatorname{Hom}_R(\beta''_j,N)} \operatorname{Hom}_R(F^{[j]},N) \xrightarrow{\operatorname{Hom}_R(\alpha''_j,N)} \operatorname{Hom}_R(F,N) \to 0$$

[1]The relations $\alpha'_j\partial = \partial\alpha'_j$ and $\alpha''_j\partial = \partial\alpha''_j$ are opposite to those in 1.1 for degree 1 chain maps.

and

$$0 \to E \otimes_R M \xrightarrow{\alpha'_j \otimes_R M} E^{[j]} \otimes_R M \xrightarrow{\beta'_j \otimes_R M} E \otimes_R M \to 0$$

whose connecting homomorphisms we denote $\eth'_j$.

For $j = 1, \dots, c$, Gulliksen defines X_j by commutativity of the squares

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M,N) & \xleftarrow{\cong} & \mathrm{H}_{-i} \operatorname{Hom}_R(F,N) \\ {\scriptstyle X_j}\downarrow & & \downarrow{\scriptstyle (-1)^i \eth'_j} \\ \operatorname{Ext}^{i+2}_R(M,N) & \xleftarrow{\cong} & \mathrm{H}_{-i-2} \operatorname{Hom}_R(F,N) \end{array}$$

with isomorphism $\mathrm{H}\operatorname{Hom}_R(F, \eta^N_R)$ and

$$\begin{array}{ccc} \mathrm{H}_i(E \otimes_R M) & \xleftarrow{\cong} & \operatorname{Tor}^R_i(L,M) \\ {\scriptstyle (-1)^i \eth'_j}\downarrow & & \downarrow{\scriptstyle X_j} \\ \mathrm{H}_{i-2}(E \otimes_R M) & \xleftarrow{\cong} & \operatorname{Tor}^R_{i-2}(L,M) \end{array}$$

with isomorphism $\mathrm{H}(E \otimes_R \varepsilon^R_M)$.

Proposition. *There are equalities* $\chi_j = X_j$ *for* $j = 1, \dots, c$.

Proof. Set $\omega(f) = (-1)^{|f|} f$, and paste (2.4.3) to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\ddot\alpha_j|_{RF}} & F^{[j]} & \xrightarrow{\ddot\beta_j|_{RF}} & F & \longrightarrow & 0 \\ & & {\scriptstyle \omega}\downarrow & & \| & & \| & & \\ 0 & \longrightarrow & F & \xrightarrow{\alpha''_j} & F^{[j]} & \xrightarrow{\beta''_j} & F & \longrightarrow & 0 \end{array}$$

along their common row. The diagram induced by applying $\operatorname{Hom}_R(-,N)$ implies the desired assertion for $\operatorname{Ext}^*_R(M,N)$. The argument for $\operatorname{Tor}^R_*(L,M)$ is similar. □

4.2. Eisenbud's construction [7, §1] begins with a *lifting* of a complex of free R–modules (F', ∂) to a pair $(\widetilde{F'}, \widetilde{\partial})$ of a graded free Q–module $\widetilde{F'}$ and its endomorphism $\widetilde{\partial}$ of degree -1, such that $(\widetilde{F'} \otimes_Q R, \widetilde{\partial} \otimes_Q R) \cong (F', \partial)$. It is easy to see that liftings always exist. Fixing one, note that $\widetilde{\partial}^2 \otimes_Q R = 0$, hence $\widetilde{\partial}^2 = \sum_{j=1}^c x_j \widetilde{t}_j(\widetilde{F'}, \widetilde{\partial})$ for some family $(\widetilde{t}_j(\widetilde{F'}, \widetilde{\partial}))_1^c$ of degree -2 endomorphisms of the graded Q–module $\widetilde{F'}$. As $\boldsymbol{\overline{x}}$ is a basis, $\widetilde{t}_j(\widetilde{F'}, \widetilde{\partial}) \otimes_Q R$ is a chain endomorphism of $\widetilde{F'} \otimes_Q R$. Let $t^{F'}_j$ be the corresponding chain endomorphism of F'.

Eisenbud defines ϵ^{left}_j for $j = 1, \dots, c$ by commutativity of the square

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M,N) & \xleftarrow{\cong} & \mathrm{H}_{-i} \operatorname{Hom}_R(F',N) \\ {\scriptstyle \epsilon^{\text{left}}_j}\downarrow & & \downarrow{\scriptstyle \mathrm{H}\operatorname{Hom}_R(t^{F'}_j, N)} \\ \operatorname{Ext}^{i+2}_R(M,N) & \xleftarrow{\cong} & \mathrm{H}_{-i-2} \operatorname{Hom}_R(F',N) \end{array}$$

and $\epsilon^{\text{right}}_j$ by that of

$$\begin{array}{ccc} \mathrm{H}_i(L \otimes_R F') & \xleftarrow{\cong} & \operatorname{Tor}^R_i(L,M) \\ {\scriptstyle \mathrm{H}(L \otimes_R t^{F'}_j)}\downarrow & & \downarrow{\scriptstyle \epsilon^{\text{right}}_j} \\ \mathrm{H}_{i-2}(L \otimes_R F') & \xleftarrow{\cong} & \operatorname{Tor}^R_{i-2}(L,M) \end{array}$$

where $F' \xrightarrow{\simeq} M$ is a free resolution of the R–module M, $\vartheta\colon F \to F'$ is a comparison of resolutions, and the isomorphisms are $\mathrm{H}\operatorname{Hom}_R(\vartheta, \eta_R^N)$ and $\mathrm{H}(\varepsilon_L^R \otimes_R \vartheta)$. Operators ϵ_j^{left} are similarly defined on $\operatorname{Tor}_i^R(L, M)$, by means of a free resolution $E' \xrightarrow{\simeq} L$.

Proposition. *On* $\operatorname{Ext}_R^*(M,N)$ *there are equalities* $\chi_j = \epsilon_j^{\text{left}}$ *for* $j = 1, \dots, c$.

On $\operatorname{Tor}_*^R(L, M)$ *there are equalities* $\chi_j = \epsilon_j^{\text{left}} = -\epsilon_j^{\text{right}}$ *for* $j = 1, \dots, c$.

Remark. This corrects a sign error in [7, Proposition 1.6], which asserts that $\epsilon_j^{\text{left}} = \epsilon_j^{\text{right}}$. The argument given there may not be used to compare the two maps: it assumes incorrectly that the canonical projections

$$E' \otimes_R M \leftarrow E' \otimes_R F' \to L \otimes_R F'$$

commute with the chain endomorphisms

$$t_j^{E'} \otimes_R M\,, \quad t_j^{E'} \otimes_R F' + E' \otimes_R t_j^{F'}\,, \quad E' \otimes t_j^{F'}\,.$$

Proof. For an R–free resolution $F' \xrightarrow{\simeq} M$, pick a homogeneous basis $\{f'_\lambda\}_{\lambda \in \Lambda}$ of the graded R–module $F'^{\natural}$, and let $P^\natural$ be a graded $K^\natural$–module on a basis $\{g_\lambda \mid |g_\lambda| = |f'_\lambda|\}_{\lambda\in\Lambda}$. Set $\omega_{<0} = 0$, and assume by induction that a κ-covariant morphism $\omega_{<h}\colon P_{<h} = \coprod_{|g_\lambda|<h} K^\natural g_\lambda \to F'_{<h}$ has been constructed such that $\omega_{<h}(g_\lambda) = f'_\lambda$ when $|g_\lambda| < h$, and $\omega_{<h}$ is a surjective quasi-isomorphism.

If $|f'_\lambda| = h$, then $\partial(f'_\lambda)$ is a cycle in $\operatorname{Im}(\omega_{<h})$, hence $\partial(f'_\lambda) = \omega_{<h}(z_\lambda)$ for some cycle $z_\lambda \in P_{<h}$. Extend the differential from $P_{<h}$ to $P_{\le h}$ by setting $\partial(g_\lambda) = z_\lambda$, and extend $\omega_{<h}$ to $\omega_{\le h}\colon P_{\le h} \to F'_{\le}$ by $\omega_{\le h}(g_\lambda) = f'_\lambda$ for all λ with $|g_\lambda| \le h$. By the choices made $\omega_{\le h}$ is a morphism and by the Five–Lemma it is a quasi-isomorphism onto its image. The induction step is complete.

Set $P = \varinjlim_h P_{<h}$ and $\omega = \varinjlim_h \omega_{<h}\colon P \to F'$.

Clearly, ω is a κ-covariant surjective morphism, and the induced map $R \otimes_K P \to F'$ is an isomorphism of complexes of R–modules. Using the notation of 2.4, we see that the pair $(P^\natural / K_+^\natural P^\natural, \partial_\varnothing)$ is a lifting of the complex F'. In view of (2.4.1) we can choose $\partial_1, \dots, \partial_c$ to be the chain endomorphisms associated with this lifting. This choice leads to equalities of Eisenbud's endomorphisms with those from 2.4:

$$t_j^{F'} = \tau_j^{F'} \qquad \text{for} \quad j = 1, \dots, c\,. \tag{4.2.1}$$

Lemma 3.1 with $\varphi = \mathrm{id}_R$ and $\gamma = \vartheta$ shows that $\vartheta\tau_j^F$ and $\tau_j^{F'}\vartheta$ are homotopic, hence

$$\mathrm{H}\operatorname{Hom}_R(\vartheta, \eta_R^N) \circ \mathrm{H}\operatorname{Hom}_R(t_j^{F'}, N) = \mathrm{H}\operatorname{Hom}_R(\tau_j^F, J) \circ \mathrm{H}\operatorname{Hom}_R(\vartheta, \eta_R^N)\,.$$

As $\mathrm{H}\operatorname{Hom}_R(\tau_j^F, J) = \chi_j$, we get $\chi_j = \epsilon_j^{\text{left}}$. The argument for Tor is similar. □

4.3. Avramov's construction [1, §1] dualizes that of Eisenbud described in 4.2.

Recall that J is the chosen injective resolution of the R–module N. Choose a graded Q–module $\widetilde{J}$ such that the Q-module $\widetilde{J}_i$ is an injective envelope of the Q–module J_i for each i. Clearly, $\operatorname{Hom}_Q(R, \widetilde{J}) \cong J$. Since $\widetilde{J}$ is a graded injective Q–module, there exits an endomorphism $\widetilde{\partial}$ of degree -1 on $\widetilde{J}$ such that $\widetilde{\partial}|_J = \partial$. By [1, Proposition 1.2] there are degree -2 endomorphisms $\widetilde{u}_j(\widetilde{J}, \boldsymbol{x})\colon \widetilde{J} \to \widetilde{J}$, which satisfy $\widetilde{\partial}^2 = \sum_{j=1}^c x_j \widetilde{u}_j(\widetilde{J}, \boldsymbol{x})$, and which restrict to degree -2 chain endomorphisms $u_j^J\colon J \to J$.

The operators $\epsilon_j^{\text{right}}$ studied in [1] are defined by commutativity of the square

$$\begin{array}{ccc} \operatorname{Ext}_R^i(M,N) & \xleftarrow{\cong} & \operatorname{H}_{-i}\operatorname{Hom}_R(M,J) \\ {\scriptstyle \epsilon_j^{\text{right}}}\downarrow & & \downarrow{\scriptstyle \operatorname{H}\operatorname{Hom}_R(M,u_j^J)} \\ \operatorname{Ext}_R^{i+2}(M,N) & \xleftarrow{\cong} & \operatorname{H}_{-i-2}\operatorname{Hom}_R(M,J) \end{array}$$

with isomorphisms $\operatorname{H}\operatorname{Hom}_R(\varepsilon_M^R, J)$.

Proposition. *On $\operatorname{Ext}_R^*(M,N)$ there are equalities $\chi_j = \epsilon_j^{\text{right}}$ for $j = 1, \dots, c$.*

Remark. Comparison with Proposition 4.2 shows that $\epsilon_j^{\text{left}} = \epsilon_j^{\text{right}}$. This is stated in [1, (1.5)], as a consequence of [1, Proposition 1.4]. However, that proposition is not correct: the morphisms $\operatorname{Hom}_R(F, \eta_R^N)$ and $\operatorname{Hom}_R(\varepsilon_M^R, J)$ do not commute with the actions induced by the chain endomorphisms t_j^F and u_j^J.

Proof. With the notation of 2.5, we set $\underline{Y} = (0\colon K_+^\natural)_Y$ and consider the endomorphisms $\partial_H\colon \underline{Y} \to \underline{Y}$ constructed in 2.5. Because $J = (0 : \mathfrak{a})_{\underline{Y}} = (0 : \mathfrak{A})_Y$ and $\partial_J = \partial_\varnothing|_J = \partial_Y|_J$, there is an injective homomorphism of graded Q–modules $\widetilde{\iota}\colon \widetilde{J} \to \underline{Y}$ with $\widetilde{\iota}|_J = \mathrm{id}_J$ and $\widetilde{\iota}\circ\widetilde{\partial} - \partial_\varnothing\circ\widetilde{\iota} = \sum_{j=1}^c x_j h_j$ for appropriate degree -1 homomorphisms of graded Q–modules $h_j : \widetilde{J} \to \underline{Y}$. A direct computation yields

$$\sum_{j=1}^c x_j \big(\widetilde{\iota}\circ\widetilde{u}_j^J(J,\boldsymbol{x}) - v_j^J\circ\widetilde{\iota}\big) = \sum_{j=1}^c x_j (h_j\circ\widetilde{\partial} + \partial_\varnothing\circ h_j)\,.$$

As the Koszul complex is exact, u_j^J is homotopic to v_j^J, yielding the middle equality

$$\epsilon_j^{\text{right}} = \operatorname{H}\operatorname{Hom}_R(F, u_j^J) = \operatorname{H}\operatorname{Hom}_R(F, v_j^J) = \chi_j\,;$$

the last one comes from Proposition 2.5. □

4.4. Mehta's construction [9, §1] proceeds from a presentation

$$0 \to B \to G_0 \xrightarrow{\varepsilon} M \to 0$$

with a projective Q–module G_0.

In view of the isomorphisms $\operatorname{Tor}_1^Q(R, G_0) = 0$ and $\operatorname{Tor}_1^Q(R, M) \cong C \otimes M$, where $C = \mathfrak{a}/\mathfrak{a}^2$ is the cotangent module, it induces an exact sequence of R–modules

$$\mathcal{E}^M\colon \quad 0 \to C \otimes M \xrightarrow{\iota} B' \to G_0' \xrightarrow{\varepsilon'} M \to 0\,.$$

where $-' = R \otimes_Q -$. In the notation of cf. 2.1, $\iota(\sum_k \overline{x}_k \otimes m_k) = 1 \otimes \sum_k x_k m_k'$, where $m_k' \in G_0$ satisfy $\varepsilon(m_k') = m_k$. For each j we get an exact sequence

$$\mathcal{E}_j^M\colon \quad 0 \to M \to M_j \to G_0' \xrightarrow{\varepsilon'} M \to 0\,.$$

as the pushout of $\mathcal{E}^M$ along $x_j^* \otimes M\colon C \otimes_R M \to R \otimes_R M = M$, where $x_1^*, \dots, x_c^*$ is the basis of C^* dual to $\overline{x}_1, \dots, \overline{x}_c$.

The iterated connecting homomorphisms induced by $\mathcal{E}_j^M$ yield operators δ_j^{right} on $\operatorname{Ext}_R^*(M,N)$ and δ_j^{left} on $\operatorname{Tor}_*^R(L,M)$ studied by Mehta. Furthermore, the exact sequences $\mathcal{E}_j^N$ and $\mathcal{E}_j^L$ yield operators δ_j^{left} on $\operatorname{Ext}_R^*(M,N)$ and δ_j^{right} on $\operatorname{Tor}_*^R(L,M)$.

Proposition. *On $\operatorname{Ext}_R^*(M,N)$ there are equalities $\chi_j = \delta_j^{\text{left}} = -\delta_j^{\text{right}}$ for $j = 1, \dots, c$.*

On $\operatorname{Tor}_*^R(L,M)$ there are equalities $\chi_j = -\delta_j^{\text{left}} = \delta_j^{\text{right}}$ for $j = 1, \dots, c$.

Remarks. (1) The last statement corrects a sign error in [9, Proposition 2.4], which asserts that $\delta_j^{\text{left}} = \delta_j^{\text{right}}$ on $\operatorname{Tor}^R_*(L, M)$.

(2) Mehta also considers an analogous exact sequence obtained as the pullback of an injective copresentation of M; when $c = 1$ he proves that these sequences are congruent with $\mathcal{E}^M_j$, and hence define the same cohomology operators. His argument in [9, Proposition 2.5] can be extended to arbitrary codimension.

Proof. Let (G, ∂) be a free resolution of M over Q. As $x_j M = 0$, for each j there is a homotopy σ^j from $x_j \operatorname{id}_G$ to 0_G. Consider the diagram of R–modules

$$\begin{array}{ccccccccccc}
G''_1 \oplus G'_3 & \xrightarrow{\partial''_3} & G''_0 \oplus G'_2 & \xrightarrow{\partial''_2} & G'_1 & \xrightarrow{\partial'_1} & G'_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle [C\otimes\varepsilon' \ 0]} & & \downarrow{\scriptstyle \pi} & & \| & & \| & & \\
0 & \longrightarrow & C \otimes M & \xrightarrow{\iota} & B' & \longrightarrow & G'_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_j^*\otimes M} & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & M_j & \longrightarrow & G'_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0
\end{array}$$

where $-'' = C \otimes_Q -$, $\partial''_3 = \begin{bmatrix} C\otimes\partial_1 & 0 \\ \sigma_1 & \partial'_3 \end{bmatrix}$ and $\partial''_2 = [\sigma_0 \ \partial'_2]$, the middle and lower rows are the exact sequences $\mathcal{E}^M$ and $\mathcal{E}^M_j$ above, π comes from the canonical factorization of ∂'_1 and $\sigma_i(\sum_k \overline{x}_k \otimes g_k) = 1 \otimes \sum_k \sigma_i^k(g_k)$ for $i = 0, 1$.

The commutativity of the upper left hand square is clear. The one to its right commutes because $\partial_1 \sigma_0^j = x_j \operatorname{id}_G$. Thus, the diagram is commutative. It is checked directly that the first row is exact[2]. We extend it to a free resolution F' of M over R, and identify $\operatorname{Ext}^*_R(M, M)$ with $\operatorname{H}\operatorname{Hom}_R(F', I)$ via the canonical isomorphism.

Let $\mu^M_j \colon F' \to I$ be the chain map which on F'_2 is the composition of $x_j^* \otimes \overline{\varepsilon}$ with the embedding $\eta \colon M \to I_0$, and is trivial elsewhere. The commutativity of the diagram implies that $-[\mu^M_j]$ is the extension class associated with the exact sequence $\mathcal{E}^M_j$, cf. [6, §7, no. 3, Definition 1]. Now [6, §7, no. 6, Corollary 3] yields

$$\delta_j^{\text{left}}(\gamma) = -\gamma \cdot [-\mu^M_j] \quad \text{and} \quad \delta_j^{\text{right}}(\gamma) = [-\mu^N_j] \cdot \gamma \quad \text{for} \quad \gamma \in \operatorname{Ext}^*_R(M, N)\,.$$

while [6, §7, no. 8, Corollaries 2 and 4] show that

$$\delta_j^{\text{right}}(t) = [-\mu^M_j] \cdot t \quad \text{and} \quad \delta_j^{\text{left}}(t) = [-\mu^L_j] \cdot t \quad \text{for} \quad t \in \operatorname{Tor}^R_*(L, M)\,.$$

On the other hand, lifting $F'_2 \to F'_1 \to F'_0$ to homomorphisms of free Q–modules

$$G_0^c \oplus G_2 \xrightarrow{\widetilde{\partial}_2} G_1 \xrightarrow{\widetilde{\partial}_1} G_0$$

with $\widetilde{\partial}_1 = \partial_1$ and $\widetilde{\partial}_2((g_1, \dots, g_c) + g) = \sum_k \sigma_0^k(g_k) + \partial_2(g)$, we see that Eisenbud's operator $t_j^{F'}$ from 4.2 acts on F'_2 by the formula $t_j^{F'}\big(\sum_k \overline{x}_k \otimes g_k + 1 \otimes g\big) = 1 \otimes g_j$. This implies $[\mu^M_j] = [\eta^M_R \circ \varepsilon^R_M \circ t_j^{F'}]$. As $t_j^{F'} = \tau_j^{F'}$ by (4.2.1), Corollary 2.4 and Theorem 3.3 yield $\chi_j = \delta_j^{\text{left}} = -\delta_j^{\text{right}}$ on $\operatorname{Ext}^*_R(M, N)$ and $\chi_j = \delta_j^{\text{right}} = -\delta_j^{\text{left}}$ on $\operatorname{Tor}^R_*(L, M)$. □

[2] In fact, it represents the beginning of a free resolution of M over R, constructed from G by Eisenbud [7], but we shall not need the rest of it.

5. Applications

We extend the important finiteness result discussed in the introduction and its recent converse to Ext's of complexes of R–modules. We also establish a new property of the cohomology operators—their primitivity.

For the first two subsections we keep the notation and hypotheses of 2.1, and as in Proposition 3.3 denote $\mathcal{S}$ the symmetric algebra of the R–module $\operatorname{Hom}_R(\mathfrak{a}/\mathfrak{a}^2, R)$, graded by assigning to its generators cohomological degree 2.

5.1. **Finiteness.** The R–module $\operatorname{Ext}^*_Q(M,N)$ is noetherian when $\operatorname{Ext}^n_Q(M,N)$ is noetherian over R (or, equivalently, over Q) for each n, and vanishes for $n \gg 0$.

Theorem. *Let M and N be complexes of R–modules. The R–module $\operatorname{Ext}^*_Q(M,N)$ is noetherian if and only if the $\mathcal{S}$–module $\operatorname{Ext}^*_R(M,N)$ is noetherian.*

Comments in place of proof. Eisenbud's constructions of operators $\epsilon^{\mathrm{left}}_j$, recalled in 4.2, can easily be extended to complexes, and the proof that they coincide with the cohomology operators χ_j carries over. Thus, by Theorem 3.3, we may consider $\operatorname{Ext}^*_R(M,N)$ as a module over the ring $\mathcal{S}$ of Eisenbud operators.

If $\operatorname{Ext}^*_Q(M,N)$ is noetherian over Q, then the spectral sequence argument given in [2] for finiteness over $\mathcal{S}$ shows that $\operatorname{Ext}^*_R(M,N)$ is noetherian over $\mathcal{S}$. (If $\mathrm{H}(M)$ is bounded below and $\mathrm{H}(N)$ is bounded above, then Gulliksen's original proof of [8, Theorem 3.1] can also be applied, in view of Proposition 4.1.)

If M and N are Q–modules and $\operatorname{Ext}^*_R(M,N)$ is noetherian over $\mathcal{S}$, then it is proved in [4, Theorem 4.2] that $\operatorname{Ext}^*_Q(M,N)$ is noetherian over Q. That argument only depends on the formal properties of the spectral sequence of [4, Theorem 4.4], whose construction works equally well for complexes. □

Let $\mathcal{Z}^*_R(M)$ denote the subalgebra of $\operatorname{Ext}^*_R(M,M)$ generated over $R/\operatorname{ann}(M)$ by the *central elements of degree* 2. From Theorems 5.1 and 3.3 we get

Corollary. *If the Q–module $\operatorname{Ext}^*_Q(M,N)$ is noetherian, then the $\mathcal{Z}^*_R(M)$–$\mathcal{Z}^*_R(N)$–bimodule $\operatorname{Ext}^*_R(M,N)$ is noetherian over each ring.* □

5.2. **Primitivity.** A DG algebra with divided powers (or DGΓ-algebra) over R is a graded commutative DG R–algebra A with $A_i = 0$ for $i \leq 0$, equipped with maps $A_{2n} \ni a \mapsto a^{(i)} \in A_{2ni}$, defined for all $n \geq 1$ and $i \geq 0$, and satisfying some standard conditions, cf. e.g. [5, (1.3)]. A *Γ-derivation* is an R–linear map $\theta\colon A \to A$ such that

$$\theta(a'a'') = \theta(a')a'' + (-1)^{|\theta||a'|}a'\theta(a'') \qquad \text{and} \qquad \theta(a^{(i)}) = \theta(a)a^{(i-1)}\,.$$

Let S and T be DGΓ algebras over R. An extension of Tate's procedure [10] of adjunction of Γ-variables to R factors the structure homomorphism $R \to S$ as the of composition $R \to E$, where $E^\natural = R\langle X\rangle$ is a free Γ-algebra over R, with a surjective quasi-isomorphism $E \xrightarrow{\simeq} S$. By [5, (2.16)], $\operatorname{Tor}^R_*(S,T) \cong \mathrm{H}_*(E \otimes_R T)$ inherits a structure of Γ-algebra, which is independent of the choice of E.

Theorem. *On $\operatorname{Tor}^R_*(S,T)$ the operator χ_j is a Γ-derivation for $1 \leq j \leq c$.*

Proof. Consider a graded Γ-algebra $\widetilde{E} = Q\langle \widetilde{X}\rangle$, with $\widetilde{X}$ a new set of divided power variables in bijective, degree preserving correspondence $\widetilde{x} \leftrightarrow x$ with those of X. This correspondence extends to an isomorphism of Γ-algebras $\widetilde{E} \otimes_Q R = E^\natural$. For each $x \in X$, choose in $\widetilde{E}$ an element x' such that $x' \otimes 1 = \partial(x)$, and let $\widetilde{\partial}$ be the unique Q–linear Γ-derivation of $\widetilde{E}$ with $\widetilde{\partial}(\widetilde{x}) = x'$. It is easy to see that $\widetilde{\partial}^2$ is a Γ-derivation.

For each $\widetilde{x} \in \widetilde{X}$ there are $e_j(\widetilde{x}) \in \widetilde{E}$ such that $\widetilde{\partial}^2(\widetilde{x}) = \sum_j x_j e_j(\widetilde{x})$. Let $\widetilde{t}_j$ be the unique Q–linear Γ-derivation of $\widetilde{E}$ such that $\widetilde{t}_j(\widetilde{x}) = e_j(\widetilde{x})$ for all $\widetilde{x} \in \widetilde{X}$. As $\widetilde{\partial}^2$ and $\sum_{j=1}^{c} x_j \widetilde{t}_j$ agree on the Γ-generators of of the Q–algebra $\widetilde{E}$, these two endomorphisms of $\widetilde{E}$ coincide. Thus, $t_j^E = \widetilde{t}_j \otimes_Q R$ is a Γ-derivation of $\widetilde{E} \otimes_Q R = E$, and hence $\mathrm{H}(t_j^E \otimes_R T)$ is a Γ-derivation of $\mathrm{Tor}_*^R(S,T)$.

As $\chi_j = \mathrm{H}(t_j^E \otimes_R T)$ by Proposition 4.2, this proves our assertion. □

We say that $\gamma \in \mathrm{Ext}_R^n(S,S)$ is *primitive* if $\gamma(tu) = 0$ whenever $t \in \mathrm{Tor}_i^R(S,S)$ and $u \in \mathrm{Tor}_{n-i}^R(S,S)$ with $0 < i < n$, and $\gamma(t^{(i)}) = 0$ for all $t \in \mathrm{Tor}_{2n}^R(S,S)$ with $n > 0$ and $i \geq 2$. Primitives form a graded R-submodule $\mathrm{Pr}_R^*(S) \subseteq \mathrm{Ext}_R^*(S,S)$, and $\mathrm{Pr}_R^2(S) = \mathrm{Ker}(\pi_R^S)$, where π_R^S is the homomorphism of R–modules

$$\pi_R^S\colon \mathrm{Ext}_R^2(S,S) \to \mathrm{Hom}_R\left(\mathrm{Tor}_1^R(S,S)^2\,,\, \mathrm{Tor}_0^R(S,S)\right), \quad \pi_R^S(\gamma)(t) = \gamma \cdot t\,.$$

We denote $\mathcal{P}_R^*(S)$ the subalgebra of $\mathrm{Ext}_R^*(S,S)$ generated over $R/\operatorname{ann}(S)$ by the *central primitive elements of degree* 2. The last theorem and Corollary 5.1 yield

Corollary. *If T is a DG Γ-algebra over R and M is a complex of R–modules such* $\mathrm{Ext}_Q^*(M,T)$ *is noetherian over Q, then* $\mathrm{Ext}_R^*(M,T)$ *is noetherian over* $\mathcal{P}_R^*(T)$. □

5.3. **CI-dimension.** Let R be a noetherian local ring with residue field k.

A *quasi-deformation* of R is a diagram of homomorphisms $R \to R' \leftarrow Q$ of local rings, with $R \to R'$ faithfully flat and $R' \leftarrow Q$ surjective with kernel generated by a regular sequence. An R–module $M \neq 0$ has *finite CI-dimension*, cf. [4], if R has a quasi-deformation such that $M \otimes_R R'$ has finite projective dimension over Q.

Theorem. *If R is a local ring with residue field k and M is a finite R–module of finite CI-dimension, then the left module* $\mathrm{Ext}_R^*(M,k)$ *is finite over the k-subalgebra* $\mathcal{P}_R^*(k) \subseteq \mathrm{Ext}_R^*(k,k)$ *generated by the central primitive elements of degree* 2.

Proof. We set $S = k \otimes_R R'$ and identify $\mathrm{Ext}_R^*(k,k) \otimes_k S$ with $\mathrm{Ext}_{R'}^*(S,S)$ through the canonical isomorphism due to the flatness of R' over R. We have

$$\mathcal{Z}_R^*(k) \otimes_k S = \mathcal{Z}_{R'}^*(S)$$

since the center of a tensor product of algebras over a field is equal to the tensor product of the centers of the factors. The flatness of R' also identifies $\mathrm{Tor}_*^R(k,k) \otimes_k S$ and $\mathrm{Tor}_*^{R'}(S,S)$ in a way compatible with products in Tor and the action of Ext, so

$$\mathrm{Pr}_R(k) \otimes_R S = \mathrm{Ker}(\pi_R^k) \otimes_R S = \mathrm{Ker}(\pi_{R'}^S) = \mathrm{Pr}_{R'}(S)$$

and we conclude that

$$\mathcal{P}_R^*(k) \otimes_k S = \mathcal{P}_{R'}^*(S)\,.$$

Identifying $\mathrm{Ext}_R^*(M,k) \otimes_k S$ with $\mathrm{Ext}_{R'}^*(M \otimes_R R', S)$, and noting that by Corollary 5.2 the latter is a noetherian module over $\mathcal{P}_{R'}^*(S)$, we conclude by flat descent that $\mathrm{Ext}_R^*(M,k)$ is noetherian over $\mathcal{P}_R^*(k)$. □

References

[1] L. L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. **96** (1989), 71–101.

[2] L. L. Avramov, R.-O Buchweitz, *Homological algebra modulo a regular sequence, with special attention to codimension two*, preprint, 1998.

[3] L. L. Avramov, H.-B. Foxby, S. Halperin, *Differential graded homological algebra*, in preparation.
[4] L. L. Avramov, V. G. Gasharov, I. V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. **86** (1997), 67-114.
[5] L. L. Avramov, S. Halperin, *Through the looking glass: A dictionary between rational homotopy theory and local algebra*, Algebra, Algebraic Topology, and their Interactions, (J.-E. Roos, ed.), Lecture Notes in Math. 1183, Springer-Verlag, Berlin, 1986, pp. 1–27.
[6] N. Bourbaki, *Algèbre*, Chapitre X. Algèbre Homologique, Masson, Paris, 1980.
[7] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), 35–64.
[8] T. H. Gulliksen, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. **34** (1974), 167–183.
[9] V. Mehta, *Endomorphisms of complexes and modules over Golod rings*, Thesis, University of California, Berkeley, Calif., 1976.
[10] J. Tate, *Homology of noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–25.

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