GORENSTEIN ALGEBRAS AND HOCHSCHILD COHOMOLOGY

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Abstract. For homomorphism $\sigma\colon K \to S$ of commutative rings, where K is Gorenstein and S is essentially of finite type and flat as a K- module, the property that all non-trivial fiber rings of σ are Gorenstein is characterized in terms of properties of the cohomology modules $\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S \otimes_K S)$.

Introduction

Each one of the main classes of commutative noetherian rings—regular, complete intersection, Gorenstein, and Cohen-Macaulay—is defined by *local* properties that require verification at *every* maximal ideal. It is therefore important to develop for these properties *global* recognition criteria involving only *finitely many* invariants. Finitely generated algebras over fields provide the test case. Our goal is to develop finitistic global tests applicable also in a more general, relative situation.

To fix notation, let K be a commutative noetherian ring and S a *flat* K-algebra *essentially of finite type*; thus, S is a homomorphic image of a localization P of a polynomial ring in d indeterminates over K. Let $\sigma\colon K \to S$ be the structure map. Following Grothendieck [14], we say that σ is Cohen-Macaulay, etc., if the ring $S \otimes_K k$ has the corresponding property for every ring homomorphism $K \to k$ with k a field. The *grade* of S over P is the smallest integer n with $\operatorname{Ext}^n_P(S, P) \neq 0$.

When $\operatorname{Spec} S$ is connected, in Theorem 6.4 we deduce from known results that σ Cohen-Macaulay implies $\operatorname{Ext}^n_P(S, P) = 0$ for $n \neq \operatorname{grade}_P S$ and follows from

$$\operatorname{Ext}^n_P(S, P) = 0 \quad \text{for} \quad \operatorname{grade}_P S < n \leq \operatorname{grade}_P S + d\,.$$

Recognizing the Gorenstein property turns out to be an entirely different matter, even when K is a Gorenstein ring. The additional information needed is that the S-module $\operatorname{Ext}^s_P(S, P)$ is cyclic for $s = \operatorname{grade}_P S$. This simple condition may be difficult to verify because the S-module structure under inspection comes through an isomorphism $\operatorname{Ext}^s_P(S, P) \cong \mathrm{H}^s(\operatorname{Hom}_P(S, I))$, where I is an injective resolution of P over itself; such a resolution incorporates input from *every* prime ideal of P.

Our results put into play the *enveloping algebra* $S^{\mathsf{e}} = S \otimes_K S$ and the S^{e}-module structure of S, given by the surjective *multiplication map* $\mu\colon S \otimes_K S \to S$. We let $\dim S$ denote the *Krull dimension* of S. When S is a domain $\operatorname{tr}\deg_K S$ denotes the *transcendence degree* of its field of fractions over that of $\sigma(K)$.

Theorem 1. *When K is Gorenstein and S is a Cohen-Macaulay domain, flat and essentially of finite type over K, the ring S is Gorenstein if and only if one has*

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) = 0 \quad \textit{for} \quad \operatorname{tr}\deg_K S < n \leq \operatorname{tr}\deg_K S + \dim S\,.$$

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When the K-module S is *projective*, rather then just flat, this result can be stated in terms of *Hochschild cohomology* by using the isomorphisms of S-modules

$$\mathrm{HH}^n(S\,|\,K;N) \cong \mathrm{Ext}^n_{S^{\mathsf{e}}}(S,N) \quad \text{for all} \quad n \in \mathbb{Z}\,.$$

Even when K is a field, Theorem 1 is new and the hypothesis on S may be hard to remove: A conjecture of Tachikawa [19, p. 115] from 1973 states that $\operatorname{rank}_K S < \infty$ and $\mathrm{HH}^n(S\,|\,K;S^{\mathsf{e}}) = 0$ for all $n \geq 1$ imply that S is Gorenstein; it is still open.

The theorem above is abstracted from Theorem 7.2. Several results that enter into its proof hold in much greater generality. First we show that Gorenstein homomorphisms can be recognized from the cohomology of a *single* module.

Theorem 2. *A flat, essentially of finite type homomorphism $\sigma\colon K \to S$ is Gorenstein if and only if the S-module $\bigoplus_{n=0}^{\infty} \mathrm{Ext}^n_{S^{\mathsf{e}}}(S,S^{\mathsf{e}})$ is projective of rank 1.*

This is contained in Theorem 2.1, whose proof depends on properties of quasi-Gorenstein homomorphisms, defined and studied in [6]. When σ is finite a new criterion for these maps is proved in Section 2. It is deduced from Theorem 1.4, which in turn strengthens a criterion of Foxby for finite Gorenstein dimension.

Theorem 2 implies that when $\operatorname{Spec} S$ is connected one has $\mathrm{Ext}^n_{S^{\mathsf{e}}}(S,S^{\mathsf{e}}) \neq 0$ for a *single* value of n. Determining that value is a key step in proving Theorem 1.

To carry it out we develop a new technology that allows us to express the modules $\mathrm{Ext}^n_{S^{\mathsf{e}}}(S,S^{\mathsf{e}})$ in terms of cohomology computed over S; it complements the classical technique of *reduction to the diagonal*. The main result in that direction is Theorem 3.1, where Gorenstein algebras appear only in a supporting role. A noteworthy aspect of its proof is that it proceeds through a long chain of quasi-isomorphisms that involve not only complexes over R and S, but also DG modules over auxiliary DG algebras. The relevant facts are reviewed in Section 8.

A useful, concrete form of Theorem 3.1 is worked out as Theorem 4.1, of which a special case is quoted below. As usual, $\Omega_{P|K}$ denotes the P-module of Kähler differentials, and $\mathbf{R}\mathrm{Hom}_P(-,-)$ the derived functor of the functor of P-linear homomorphisms. The complex $D^M = \mathbf{R}\mathrm{Hom}_P(M, \bigwedge^d_P \Omega_{P|K})$ is well-defined in the derived category of S-modules; it carries an action of S induced by that on M.

Theorem 3. *Let $K \to P \to S$ be a factorization of the map $\sigma\colon K \to S$, where P is a localization of a polynomial ring in d variables over K and S is finite over P.*

If M and N are S-modules, such that M is flat over K and finite over S, then for each $n \in \mathbb{Z}$ there is an isomorphism of S-modules

$$\mathrm{Ext}^n_{S^{\mathsf{e}}}(S, M \otimes_K N) \cong \mathrm{Ext}^{n-d}_S(\mathbf{R}\mathrm{Hom}_P(M, \textstyle\bigwedge^d_R \Omega_{P|K}), N)\,.$$

Here we apply Theorem 3 with $M = S = N$. Its first use is to prove non-vanishing results about Hochschild cohomology; from Theorem 5.1 we quote:

Theorem 4. *If $\sigma\colon K \to S$ is a flat homomorphism essentially of finite type and $\mathfrak{q}$ is a minimal prime ideal of S, then one has*

$$\mathrm{Ext}^t_{S^{\mathsf{e}}}(S,S^{\mathsf{e}})_{\mathfrak{q}} \neq 0 \quad \textit{for} \quad t = \operatorname{tr\,deg}_K(S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}})\,.$$

Theorems 2 and 4 imply the ‘only’ if part of Theorem 1. For the converse we use the relative dualizing complex $D^S = \mathbf{R}\mathrm{Hom}_P(S, \bigwedge^d_P \Omega_{P|K})$. Its homology module

$$C^S = \mathrm{H}^s(D^S) \quad \text{where} \quad s = \operatorname{grade}_P S$$

appears in the next result, which incorporates parts of Theorems 6.1 and 6.2.

Theorem 5. *If σ is Cohen-Macaulay and* $\operatorname{Spec} S$ *is connected, then* $\operatorname{tr}\deg_K S_{\mathfrak q}/\mathfrak q S_{\mathfrak q}$ *has the same value, say t, for every minimal prime ideal $\mathfrak q$ of S.*

For each $n \in \mathbb{Z}$ there is an isomorphism of S-modules

$$\operatorname{Ext}^n_{S^{\mathsf e}}(S, S^{\mathsf e}) \cong \operatorname{Ext}^{n-t}_S(C^S, S)\,.$$

When K is Gorenstein $C^S_{\mathfrak n}$ is a canonical module for $S_{\mathfrak n}$ for each $\mathfrak n \in \operatorname{Spec} S$.

To finish the proof of Theorem 1 we invoke a recent result, proved independently in [3] and [15]: $\operatorname{Ext}^n_S(C^S, S) = 0$ for $1 \le n \le \dim S$ implies that S is Gorenstein.

The characterizations of the Gorenstein property discussed above resonate with earlier descriptions of smoothness in similar terms; see 7.3. The mathematics behind this observation is still work in progress, but a philosophical explanation can be advanced: Homological properties of a flat algebra, viewed as a bimodule over itself, encode information about all its fibers. The code is similar to the one that translates properties of a local ring into homological data on its residue field.

1. Homological dimensions

In this paper rings are commutative. For modules *finite* means finitely generated. The results in this section concern an invariant of complexes called G-dimension (for *Gorenstein* dimension), defined for finite modules by Auslander and Bridger [2].

Let R be a commutative ring. We write $\mathsf{D}(R)$ for the derived category of R-modules. It objects are the complexes of R-modules

$$\cdots \to M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \to \cdots$$

We write $M \xrightarrow{\simeq} N$ to denote a morphism of complexes that induces an isomorphism in homology, and call it a *quasi-isomorphism*. The notation $M \simeq N$ indicates that that there is a chain of quasi-isomorphisms linking M to N; that is, M and N are isomorphic in $\mathsf{D}(R)$. Section 8 contains more details on homological algebra.

A complex M is *homologically finite* if the graded R-module $\mathrm{H}(M)$ is finite.

1.1. Let R be a noetherian ring.

An R-module G is said to be *totally reflexive* if it satisfies

$$\begin{aligned} G \cong& \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R) \quad \text{and} \\ \operatorname{Ext}^n_R(G,R) = 0 =& \operatorname{Ext}^n_R(\operatorname{Hom}_R(G,R),R) \quad \text{for all} \quad n \ge 1\,. \end{aligned}$$

The *G-dimension* of a homologically finite complex M of R-modules is the number

$$\operatorname{G-dim}_R M = \inf_n \left\{ n \ge \sup \mathrm{H}(M) \;\middle|\; \begin{matrix} \operatorname{Coker}(\partial^P_{n+1}) \text{ is totally reflexive in some} \\ \text{semiprojective resolution } P \xrightarrow{\simeq} M \end{matrix} \right\}.$$

Finite projective modules are totally reflexive, so $\operatorname{G-dim}_R M \le \operatorname{pd}_R M$. Every finite module (equivalently, homologically finite complex) has finite G- dimension if and only if the ring R is Gorenstein; see [2, (4.20)].

1.2. Foxby [11, (2.2.3)] has obtained an alternative characterization of complexes of finite G-dimension: A homologically finite complex of R-modules M has finite G-dimension if and only the following conditions hold.

(a) The canonical *biduality map* in $\mathsf{D}(R)$ is an isomorphism:

$$\delta^M \colon M \xrightarrow{\simeq} \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,R),R)\,.$$

(b) The complex $\mathbf{R}\operatorname{Hom}_R(M,R)$ is homologically bounded.

Moreover, when these conditions hold one has

$$\text{G-dim}_R M = \sup\{n \mid \operatorname{Ext}^n_R(M,R) \neq 0\}\,. \tag{1.2.1}$$

Some classical ring-theoretic invariants provide bounds for G- dimensions.

1.3. The *grade* of a finite R-module M is the integer

$$\operatorname{grade}_R M = \inf\{n \mid \operatorname{Ext}^n_R(M,R) \neq 0\}\,. \tag{1.3.1}$$

It equals the maximal length of an R-regular sequence in $\operatorname{Ann}_R M$, the annihilator of M. Krull's Principal Ideal Theorem yields one of the inequalities below:

$$\operatorname{grade}_R M \leq \min\{\operatorname{height} \operatorname{Ann}_R M\,,\ \text{G-dim}_R M\}\,. \tag{1.3.2}$$

For the other one it suffices to compare formulas (1.3.1) and (1.2.1).

We provide a more flexible version of Foxby's characterization of finite Gorenstein dimension in 1.2. We write $\operatorname{Max} R$ for the set of maximal ideals of R.

Theorem 1.4. *Let R be a noetherian ring and M a complex with $\mathrm{H}(M)$ finite.*

The complex M then has finite G-dimension when the following conditions hold:

(a) *For each maximal ideal $\mathfrak{m}$ in R, in $\mathsf{D}(R_\mathfrak{m})$ there exists an isomorphism*

$$M_\mathfrak{m} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}), R_\mathfrak{m})\,.$$

(b) *The complex $\mathbf{R}\mathrm{Hom}_R(M,R)$ is homologically bounded or $\dim R$ is finite.*

As a corollary we obtain a homological test for finite G-dimension.

Corollary 1.5. *Let M be a finite R-module.*

If for each $\mathfrak{m} \in \operatorname{Max} R$ there exists an integer $d(\mathfrak{m})$, such that one has

$$\operatorname{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \cong \begin{cases} \Sigma^{d(\mathfrak{m})} M_\mathfrak{m} & \text{for} \quad n = d(\mathfrak{m})\,; \\ 0 & \text{for} \quad n \neq d(\mathfrak{m})\,, \end{cases}$$

then the following inequalities hold:

$$\text{G-dim}_R M \leq \sup\left\{\operatorname{height}\mathfrak{p} \,\middle|\, \begin{matrix} \mathfrak{p} \text{ a prime ideal in } R \\ \text{minimal over } \operatorname{Ann}_R M \end{matrix}\right\} < \infty\,.$$

Proof. The hypothesis implies $\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \simeq \Sigma^{d(\mathfrak{m})} M_\mathfrak{m}$, and hence one has

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}), R_\mathfrak{m}) &\simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(\Sigma^{d(\mathfrak{m})} M_\mathfrak{m}, R_\mathfrak{m}) \\ &\simeq \Sigma^{-d(\mathfrak{m})}\, \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \\ &\simeq M_\mathfrak{m} \end{aligned}$$

in $\mathsf{D}(R_\mathfrak{m})$. In particular, setting

$$h = \max\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ is minimal over } \operatorname{Ann}_R M\}$$

from (1.3.1), and (1.3.2) one obtains inequalities

$$d(\mathfrak{m}) = \operatorname{grade}_{R_\mathfrak{m}} M_\mathfrak{m} \leq \operatorname{height}(\operatorname{Ann}_{R_\mathfrak{m}} M_\mathfrak{m}) \leq h\,.$$

They imply $\operatorname{Ext}^n_R(M,R) = 0$ for $n > h$. Thus, $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(M,R))$ is bounded, so one gets $\text{G-dim}_R M < \infty$ from Theorem 1.4, then $\text{G-dim}_R M \leq h$ from (1.2.1). □

The proof of the theorem uses a result of independent interest:

Proposition 1.6. *Let R be local ring and X a complex with the R-modules $\mathrm{H}_i(X)$ finite. If $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(X,R)) = 0$ for $|i| \gg 0$, then $\mathrm{H}_i(X) = 0$ for $i \ll 0$.*

Proof. Let k be the residue field of R. As $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(X,R))$ is bounded, one has

$$\mathrm{H}_n(\mathbf{R}\mathrm{Hom}_R(k,\mathbf{R}\mathrm{Hom}_R(X,R)))=0 \quad\text{for}\quad n\gg 0\,.$$

On the other hand, adjunctions yield the first two isomorphisms below:

$$\begin{aligned}\mathrm{H}_n(\mathbf{R}\mathrm{Hom}_R(k,\mathbf{R}\mathrm{Hom}_R(X,R))) &\cong \mathrm{H}_n(\mathbf{R}\mathrm{Hom}_R(k\otimes^{\mathbf{L}}_R X,R))\\ &\cong \mathrm{H}_n(\mathbf{R}\mathrm{Hom}_k(k\otimes^{\mathbf{L}}_R X,\mathbf{R}\mathrm{Hom}_R(k,R)))\\ &\cong \mathrm{H}_n(\mathrm{Hom}_k(\mathrm{H}(k\otimes^{\mathbf{L}}_R X),\mathrm{Ext}_R(k,R)))\\ &= \prod_{j+i=-n}\mathrm{Hom}_k(\mathrm{H}_i(k\otimes^{\mathbf{L}}_R X),\mathrm{Ext}^j_R(k,R))\,.\end{aligned}$$

As $\mathrm{Ext}_R(k,R)\neq 0$, it follows that one has $\mathrm{H}_i(k\otimes^{\mathbf{L}}_R X)=0$ for $i\ll 0$. At this point, one can conclude that $\mathrm{H}_i(X)=0$ holds for $i\ll 0$ by invoking [13, (4.5)]. What follows is a direct and elementary proof, which exploits an idea from [12, (5.12)].

The first step is to verify that for any complex C for which the length of $\mathrm{H}(C)$ is finite, one has $\mathrm{H}_i(C\otimes_R X)=0$ for $i\ll 0$. In the case where C is a module, this is achieved by an obvious induction on the length. The general case is settled by induction on the number of non-zero homology modules of C.

Let now C be the Koszul complex on a subset $\boldsymbol{x}=\{x_1,\dots,x_e\}$ of R. Since the length of $\mathrm{H}(C)$ is finite, one has $\mathrm{H}_i(C\otimes_R X)=0$ for $i\ll 0$. Recall that C is equal to $C'\otimes_R C''$, where C' and C'' are Koszul complexes on $\{x_1,\dots,x_{e-1}\}$ and x_e, respectively. Thus, one has an exact sequence of complexes

$$0\to C'\otimes_R X\to C\otimes_R X\to \Sigma(C'\otimes_R X)\to 0\,.$$

Its homology exact sequence yields exact sequences of R-modules

$$0\longrightarrow \mathrm{H}_i(C'\otimes_R X)/x_e\,\mathrm{H}_i(C'\otimes_R X)\longrightarrow \mathrm{H}_i(C\otimes_R X)\longrightarrow \mathrm{H}_{i-1}(C'\otimes_R X)$$

By induction on e, one deduces that each R-module $\mathrm{H}_i(C\otimes_R X)$ is finite.

For the final step, choose $\boldsymbol{x}$ to generate the maximal ideal of R. The length $\mathrm{H}(C)$ then is finite, so the first step yields $\mathrm{H}_i(C\otimes_R X)=0$ for $i\ll 0$. The exact sequence above then gives $\mathrm{H}_n(C'\otimes_R X)/x_e\,\mathrm{H}_n(C'\otimes_R X)=0$ for $i\ll 0$, which implies $\mathrm{H}_i(C'\otimes_R X)=0$ for $i\ll 0$, by Nakayama's Lemma. Splitting off one element x_j at a time, we arrive at $\mathrm{H}_i(X)=0$ for $i\ll 0$, as desired. $\square$

Proof of Theorem 1.4. Set $(-)^*=\mathbf{R}\mathrm{Hom}_R(-,R)$. We start with the local case:

Claim. Assume that R is a local ring. It M is a homologically bounded complex, and there exits an isomorphism $\mu\colon M\to M^{**}$ in $\mathsf{D}(R)$, then $\text{G-dim}_R\,M$ is finite.

Indeed, as $\mathrm{H}(M)$ is bounded one has $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M,R))=0$ for $i\gg 0$. Moreover, the R-module $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M,R))$ is finite for each i, so the isomorphism μ and Proposition 1.6 yield $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M,R))=0$ for $i\ll 0$. It thus remains to prove that the biduality morphism $\delta_M\colon M\to M^{**}$ is an isomorphism; see 1.2.

The composition $(\delta_M)^*\delta_{M^*}$ is the identity map of M^*, so for each $n\in\mathbb{Z}$ the map $\mathrm{H}_n(\delta_{M^*})\colon\ \mathrm{H}_n(M^*)\to\mathrm{H}_n(M^{***})$ is a split monomorphism. On the other hand, μ induces an isomorphism $\mathrm{H}_n(M^{***})\cong\mathrm{H}_n(M^*)$ for each $n\in\mathbb{Z}$. As the R-module $\mathrm{H}_n(M^*)=\mathrm{Ext}^{-n}_R(M,R)$ is finite, we conclude that $\mathrm{H}_n(\delta_{M^*})$ is bijective. Thus,

δ_{M^*} is an isomorphism in $\mathsf{D}(R)$, and hence so is $\delta_{M^{**}}$. Since the square

$$\begin{tikzcd} M \arrow[r, "\mu", "\simeq"'] \arrow[d, "\delta_M"'] & M^{**} \arrow[d, "\delta_{M^{**}}", "\simeq"'] \\ M^{**} \arrow[r, "\mu^{**}", "\simeq"'] & M^{****} \end{tikzcd}$$

in $\mathsf{D}(R)$ commutes, we see that δ_M is an isomorphism, as desired.

This completes the proof of the claim.

At this point, we can conclude that when condition (a) holds, the number $\operatorname{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}$ is finite for each $\mathfrak{m} \in \operatorname{Max} R$. When $\mathbf{R}\mathrm{Hom}_R(M,R)$ is homologically bounded, it is homologically finite, so one has $(\delta_M)_\mathfrak{m} = \delta_{M_\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Max} R$. As $\delta_{M_\mathfrak{m}}$ is an isomorphism, so is δ_M, that is to say, $\operatorname{G-dim}_R M$ is finite; see 1.2.

It remains to prove the theorem when $\dim R$ is finite. Since $\operatorname{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}$ is finite for each $\mathfrak{m} \in \operatorname{Max} R$, (1.2.1) gives the second equality below:

$$\begin{aligned} -\inf \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(M,R))_\mathfrak{m} &= -\inf \mathrm{H}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m},R_\mathfrak{m})) \\ &= \operatorname{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m} \\ &= \operatorname{depth} R_\mathfrak{m} - \operatorname{depth} M_\mathfrak{m} \\ &\le \dim R_\mathfrak{m} + \sup \mathrm{H}(M_\mathfrak{m}) \\ &\le \dim R + \sup \mathrm{H}(M) \end{aligned}$$

The third one is the Auslander-Bridger Equality [11, (2.3.13)], where the notion of depth for complexes is taken from [11, §A.6]. The first inequality follows from [11, (A.6.1.1)], and the other inequality is evident. Thus, $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M,R))$ vanishes for $i \ll 0$, so part (a) applies and $\operatorname{G-dim}_R M$ is finite.

This completes the proof of the theorem. □

2. QUASI-GORENSTEIN HOMOMORPHISMS

In this section $\sigma\colon K \to S$ denotes a homomorphism of rings.

One says that σ is of *finite type* if it admits a factorization $K \xrightarrow{\varkappa} P \xrightarrow{\pi} S$, where P is a polynomial ring $K[x_1,\dots,x_d]$, $\varkappa$ is the canonical homomorphism, and π is a surjective homomorphism. A wider class of homomorphisms, those *essentially of finite type*, is defined by allowing P to be a localization of $K[x_1,\dots,x_d]$.

Assume that σ is flat. We say that σ is *Gorenstein at* some $\mathfrak{n} \in \operatorname{Spec} S$ if the local ring $S_\mathfrak{n}/(\mathfrak{n}\cap K)S_\mathfrak{n}$ is Gorenstein. The homomorphism σ is *Gorenstein* when this condition holds for every $\mathfrak{n} \in \operatorname{Spec} S$; equivalently, the fiber rings are Gorenstein; recall that the *fiber ring* of σ at $\mathfrak{p} \in \operatorname{Spec} K$ is $k(\mathfrak{p}) \otimes_K S$, where $k(\mathfrak{p}) = K_\mathfrak{p}/\mathfrak{p}K_\mathfrak{p}$.

We establish new characterizations of Gorenstein homomorphisms in terms of the *product map* $\mu\colon S^\mathsf{e} \to S$, where $S^\mathsf{e} = S \otimes_K S$ and $\mu(a \otimes b) = ab$.

Theorem 2.1. *Let K be a noetherian ring and let $\sigma\colon K \to S$ be a flat, essentially of finite type homomorphism of rings.*

The following conditions are then equivalent.

- (i) *The homomorphism σ is Gorenstein.*
- (i′) *The homomorphism $\sigma \otimes_K S$ is Gorenstein at every $\mathfrak{m} \supseteq \operatorname{Ker}(\mu)$.*
- (ii) *The homomorphism $\mu\colon S^\mathsf{e} \to S$ is quasi-Gorenstein.*
- (iii) *The S^e-module S has finite G-dimension.*
- (iv) *The graded S-module $\operatorname{Ext}_{S^\mathsf{e}}(S,S^\mathsf{e})$ is projective of rank 1.*

A detailed version of condition (iv) can be found in Theorem 7.1. To explain condition (ii), and for further use, we introduce some terminology.

2.2. A *factorization* of σ is an equality $\sigma = \pi\varkappa$ where $\varkappa$ and π are homomorphisms of rings, with $\varkappa$ flat and π finite. We say that such a factorization is surjective, or local, if π has the corresponding property. Similarly, we say that it is (essentially) of finite type, or Gorenstein, or smooth, if $\varkappa$ is. Recall that (essentially) smooth means (essentially) of finite type, flat, and with geometrically regular fibers.

Any homomorphism σ (essentially) of finite type has a surjective (essentially) smooth factorization. Indeed, one has $\sigma = \pi\varkappa$ with π surjective and $\varkappa$ the flat map $K \to U^{-1}K[x_1, \dots, x_d]$; for each $\mathfrak{p} \in \operatorname{Spec} K$ the fiber of π is a localization of the polynomial ring $k(\mathfrak{p})[x_1, \dots, x_d]$, and so is a geometrically regular $k(\mathfrak{p})$-algebra.

A homomorphism of rings is *local* if its source and target are local rings, and it maps the unique maximal ideal of the source into that of the target. The *localization* of σ at a prime ideal $\mathfrak{n}$ of S is the induced local homomorphism $\sigma_{\mathfrak{n}} \colon K_{\mathfrak{n}\cap K} \to S_{\mathfrak{n}}$.

2.3. Assume that $\sigma \colon K \to S$ is essentially of finite type. We say that it has *finite G-dimension* at some $\mathfrak{n} \in \operatorname{Spec} S$, and write $\operatorname{G-dim} \sigma_{\mathfrak{n}} < \infty$, if for some surjective Gorenstein factorization $K \xrightarrow{\varkappa} P \xrightarrow{\pi} S$ one has $\operatorname{G-dim}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} < \infty$. This property does not depend on the choice of factorization; see [6, (4.3)].

When $\operatorname{G-dim}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ is finite the complex $\mathbf{R}\mathrm{Hom}_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P})$ does not depend on factorizations, up to isomorphism and shift in $\mathsf{D}(S_{\mathfrak{n}})$; see [6, (6.6), (6.7), (5.5)].

If the map σ is flat, or if the ring K is Gorenstein, then the homomorphism σ has finite G-dimension; see [6, (4.4.1), (4.4.2)].

2.4. We say that σ is *quasi-Gorenstein* at $\mathfrak{n}$ if it has finite G-dimension at $\mathfrak{n}$, and $\mathbf{R}\mathrm{Hom}_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P})$ is isomorphic in $\mathsf{D}(S_{\mathfrak{n}})$ to a shift of $S_{\mathfrak{n}}$. When σ satisfies this condition at each $\mathfrak{n} \in \operatorname{Spec} S$, we say that it is *quasi-Gorenstein*; see [6, (7.8)].

When the map σ is flat, it is quasi-Gorenstein at $\mathfrak{n}$ if and only if it is Gorenstein at $\mathfrak{n}$; see [6, (8.1)]. When the ring K is Gorenstein, the homomorphism σ is quasi-Gorenstein if and only if the ring S is Gorenstein; see [6, (8.2)].

2.5. Recall that a finite S-module L is projective if and only if for each $\mathfrak{n} \in \operatorname{Spec} S$ the $S_{\mathfrak{n}}$-module $L_{\mathfrak{n}}$ is free. Let L be such a module. On $\operatorname{Spec} S$ the function

$$\mathfrak{n} \mapsto \operatorname{rank}_{S_{\mathfrak{n}}} L_{\mathfrak{n}}$$

is upper semi-continuous, and hence constant on each connected component; if every $\mathfrak{n} \in \operatorname{Spec} S$ has $\operatorname{rank}_{S_{\mathfrak{n}}} L_{\mathfrak{n}} = d$ one says that L has *rank* d.

We say that a finite graded S-module $L = (L^i)_{i\in\mathbb{Z}}$ is projective (of rank d) if the S-module $\bigoplus_{i\in\mathbb{Z}} L^i$ is projective (of rank d).

Theorem 2.6. *Let $\varphi \colon R \to S$ be a finite homomorphism of noetherian rings.*

If the graded S-module $\operatorname{Ext}_R(S, R)$ is projective of rank 1, then $\operatorname{G-dim}_R S$ is finite and φ is quasi-Gorenstein.

Proof. For each $\mathfrak{n} \in \operatorname{Max} S$ the finiteness of S and the hypothesis on φ yield

$$\operatorname{Ext}_{R_{\mathfrak{n}\cap R}}(S_{\mathfrak{n}}, R_{\mathfrak{n}\cap R}) \cong \operatorname{Ext}_R(S, R)_{\mathfrak{n}} \cong \Sigma^{d(\mathfrak{n})} S_{\mathfrak{n}}$$

for some integer $d(\mathfrak{m})$. From Corollary 1.5 one then gets $\operatorname{G-dim}_R S < \infty$ and $\operatorname{G-dim}_{R_{\mathfrak{n}\cap R}} S_{\mathfrak{n}} < \infty$. It follows that φ is Gorenstein at $\mathfrak{n}$; see 2.4 □

The finiteness hypothesis on ψ in the next result can be dropped if defines finite G-dimension and quasi-Gorenstein homomorphism as in [6]; the same proof applies.

Corollary 2.7. *If Q is a noetherian ring, $\psi\colon Q \to R$ a homomorphism essentially of finite type, and $\varphi\psi$ is quasi-Gorenstein, then the following are equivalent.*

(i) *ψ is quasi-Gorenstein at every $\mathfrak{m} \supseteq \operatorname{Ker}(\varphi)$.*
(ii) *φ is quasi-Gorenstein and $\operatorname{G-dim}\psi_{\mathfrak{m}}$ is finite for every $\mathfrak{m} \supseteq \operatorname{Ker}(\varphi)$*
(iii) *$\operatorname{G-dim}_R S$ is finite and $\operatorname{G-dim}\psi_{\mathfrak{m}}$ is finite for every $\mathfrak{m} \supseteq \operatorname{Ker}(\varphi)$.*

Proof. The theorem shows that (ii) implies (iii). The implications (iii) $\implies$ (i) and (i) $\implies$ (ii) come from [6, (8.10)(b) and (8.10)(a)], respectively. ☐

Proof of Theorem 2.1. (i) $\implies$ (i$'$). Set $\psi = S\otimes_K\sigma\colon S \to S^{\mathsf{e}}$. This homomorphism is flat along with σ, and hence so is $\psi_{\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Spec} S^{\mathsf{e}}$; in particular, $\operatorname{G-dim}\psi_{\mathfrak{m}}$ is finite. Setting $\mathfrak{n} = \mathfrak{m} \cap S$, note the isomorphism of rings

$$k(\mathfrak{n}) \otimes_S S^{\mathsf{e}} \cong k(\mathfrak{n}) \otimes_{k(\mathfrak{m}\cap Q)} (k(\mathfrak{m} \cap Q) \otimes_Q S)\,.$$

It shows that when $k(\mathfrak{n} \cap Q) \otimes_Q S$ is Gorenstein so is $k(\mathfrak{n}) \otimes_S S^{\mathsf{e}}$, hence also $(k(\mathfrak{m}) \otimes_S S^{\mathsf{e}})_{\mathfrak{n}}$, which is isomorphic to $k(\mathfrak{n}) \otimes_{S_{\mathfrak{m}}} S^{\mathsf{e}}_{\mathfrak{m}}$.

(i$'$) $\implies$ (i). Note that every $\mathfrak{n} \in \operatorname{Spec} S$ is the contraction of the prime ideal $\mathfrak{m} = \mu^{-1}(\mathfrak{n})$ os S^{e}, which contains $\operatorname{Ker}(\mu)$. Thus, when (i$'$) holds the local homomorphism $\rho_{\mathfrak{m}}$ is flat with Gorenstein closed fiber. The *proof* of [5, (6.6)] shows that then so is the local homomorphism $\sigma_{\mathfrak{m}}\colon K_{\mathfrak{m}\cap K} \to S_{\mathfrak{m}}$.

The homomorphism ρ is essentially of finite type and satisfies $\mu\rho = \mathrm{id}^S$. As id^S is quasi-Gorenstein, Corollary 2.7 establishes the equivalence of conditions (i$'$), (ii), and (iii). Conditions (ii) and (iv) are equivalent by Theorem 2.6. ☐

3. Gorenstein factorizations

This section is devoted to one of the main results of the paper, a general reduction theorem for Hochschild cohomology with coefficients in 'tensor-decomposable' bimodules. What makes it an efficient tool in computation is the fact that Gorenstein factorizations always exist, see 2.2, and that for every flat Gorenstein K-algebra P the P-module $\operatorname{Ext}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}})$ is projective of rank 1, see Theorem 2.1. In contrast to the proof of that result, which depends on a substantial amount of commutative ring theory, the arguments in this section are purely homological.

Theorem 3.1. *Let K be a noetherian ring, $\sigma\colon K \to S$ a homomorphism of rings, $K \xrightarrow{\varkappa} P \xrightarrow{\pi} S$ a Gorenstein factorization of σ, and set*

$$W = \operatorname{Hom}_P(\operatorname{Ext}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}), P)\,.$$

If M and N are complexes of S-modules of finite flat dimension over P, and the S-module $\operatorname{H}(M)$ is finite, then in $\mathsf{D}(S)$ there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_{S^{\mathsf{e}}}(S, M \otimes^{\mathbf{L}}_K N) \simeq \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_P(M, W), N)$$

of complexes of S-modules, which is natural in both M and N.

The theorem is proved through a long computation, which takes up the rest of the section. To indicate its pedigree, and to explain what kind of complications will arise, we first discuss a very degenerate special case.

Remark 3.2. When P is equal to K and M is a finite projective S-module, the theorem yields an isomorphism of graded S-modules, which can be recovered by a simple calculation, as in [19, p. 114]; it only uses standard isomorphisms, see 8.1:

$$\begin{aligned}\operatorname{Ext}_{S^{\mathsf e}}(S, M \otimes_K N) &\cong \operatorname{Ext}_{S^{\mathsf e}}(S, \operatorname{Hom}_K(\operatorname{Hom}_K(M,K),N))\\ &\cong \operatorname{Ext}_S(\operatorname{Hom}_K(M,K),N)\end{aligned}$$

Such an argument does not extend, for M need not be finite over K. The strategy in the general case is to replace S by a suitable Differential Graded (henceforth abbreviated to DG) algebra. The difficulty then arises in keeping track of the various finiteness hypotheses in the change of DG algebras and DG modules.

Proof of Theorem 3.1. We proceed through series of quasi-isomorphisms, breaking up the argument into a sequence of steps and introducing notation as we go.

We start by framing a home for the morphisms of auxiliary DG algebras:

$$\begin{tikzcd}
 & & K \arrow[dl] \arrow[dr] & & \\
 & P^{\mathsf e} \arrow[r] \arrow[dl] & A \arrow[r, two heads, "\simeq", "\alpha"'] \arrow[dl] & P \arrow[dl] \arrow[dr] & \\
B^{\mathsf e} \arrow[r] \arrow[dddd, two heads, "\simeq"', "\beta^{\mathsf e}"] & B^{\mathsf e} \otimes_{P^{\mathsf e}} A \arrow[r, two heads, "\simeq"] \arrow[d, equal] & B^{\mathsf e} \otimes_{P^{\mathsf e}} P \arrow[r, two heads, "\cong"] & B \otimes_P B \arrow[r, two heads] \arrow[d, equal] & B \arrow[dddd, two heads, "\beta"', "\simeq"] \\
 & C \arrow[rr, two heads, "\simeq", "\gamma"'] & & D & \\
 & & & & \\
 & & & & \\
S^{\mathsf e} \arrow[rrrr, two heads] & & & & S
\end{tikzcd}$$

Arrows tagged with $\simeq$ are quasi-isomorphisms; those tipped with $\twoheadrightarrow$ are surjections.

The composition in the top row is equal to $P \otimes_K P \to P$, and A is a DG algebra with each A_i a finite projective $P^{\mathsf e}$-module, and with $A_i = 0$ for $i < 0$; see 8.5.

The composition on right-hand side is equal to $\pi\colon P \to S$, and B is a DG algebra with each B_i a finite projective P-module and $B_i = 0$ for $i < 0$; again, see 8.5.

The maps on the left-hand side are induced from those on the right hand side.

The parallelograms are base changes along the morphism $P^{\mathsf e} \to B^{\mathsf e}$.

The equalities define D, C, and γ.

The remaining maps are canonical.

It is clear from the construction that the diagram is commutative. To see that $B^{\mathsf e} \to S^{\mathsf e}$ is a quasi-isomorphism, factor it as $B^{\mathsf e} \to S \otimes_K B \to S^{\mathsf e}$ and note that each map in the composition is a tensor product of $B \xrightarrow{\simeq} S$ with a bounded below complex of flat K-modules. To account for the quasi-isomorphisms in the middle note that $B^{\mathsf e}$ is a bounded below complex of projective $P^{\mathsf e}$-modules.

Next we outline the structure of the argument.

In the course of the proof we build DG modules over various DG algebras appearing in the diagram above. The DG algebra involved is always indicated, but not so the specific arguments of functors of homomorphisms and tensor products, through which it operates; a diligent reader should have no trouble identifying these from the context. One needs to verify that all quasi-isomorphisms used commute with the corresponding actions; this is easy, as most maps are canonical. We always specify the DG algebra over which a given map, or sequence of maps, is linear; in doing this, we usually chose an algebra whose action is easy to identify. These DG algebras change as the argument proceeds, but the diagram provides a morphisms from B^{e} to each one of them, so all the morphisms involved are B^{e}-linear.

Step 1. By abuse of notation, let W also denote the complex of P-modules with underlying graded module $W = \operatorname{Hom}_P(\operatorname{Ext}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}}), P)$ and zero differential.

Let $W \xrightarrow{\simeq} I$ be a semiinjective resolution over P; see 8.2.

Let $U \xrightarrow{\simeq} M$ be a semiprojective resolution over B, with each U_i a finite projective P-module; see 8.6.

Let $U' \xrightarrow{\simeq} \operatorname{Hom}_P(U, W)$ be a semiprojective resolution over B; see 8.2.

Let $N \xrightarrow{\simeq} J$ be a semiinjective resolution over B; see 8.2.

For $J' = \operatorname{Hom}_B(S, J)$ there exists then a canonical decomposition

$$N \xrightarrow{\simeq} J' \xrightarrow{\simeq} J\,,$$

where the first map is a semiinjective resolution over S.

Furthermore, there is a quasi-isomorphism of DG B-modules

$$\operatorname{Hom}_S(\operatorname{Hom}_P(M, I), J') \simeq \operatorname{Hom}_B(U', J)\,.$$

Proof. The map $N \xrightarrow{\simeq} J$ induces the vertical arrows in the commutative diagram

$$\begin{tikzcd}
N \arrow[r, equal] \arrow[d, dotted] & \operatorname{Hom}_B(S,N) \arrow[r, "\cong", "\operatorname{Hom}_B(\beta,N)"'] \arrow[d] & \operatorname{Hom}_B(B,N) \arrow[d, "\simeq"] \\
J' \arrow[r, equal] & \operatorname{Hom}_B(S,J) \arrow[r, tail, "\simeq", "\operatorname{Hom}_B(\beta,J)"'] & \operatorname{Hom}_B(B,J)
\end{tikzcd}$$

As B acts on N through β, which is surjective map $\operatorname{Hom}_B(\beta, N)$ is bijective; the map $\operatorname{Hom}_B(\beta, J)$ is a quasi-isomorphism because β is one and J is semiinjective. By 8.3(2), J' is semiinjective, so $N \to J'$ is a semiinjective resolution over S.

One now has a sequence of standard quasi-isomorphisms of DG B-modules

$$\begin{aligned}
\operatorname{Hom}_S(\operatorname{Hom}_P(M, I), J') &\cong \operatorname{Hom}_B(S \otimes_B \operatorname{Hom}_P(M, I), J) \\
&\cong \operatorname{Hom}_B(\operatorname{Hom}_P(M, I), J) \\
&\simeq \operatorname{Hom}_B(\operatorname{Hom}_P(U, I), J) \\
&\simeq \operatorname{Hom}_B(\operatorname{Hom}_P(U, W), J) \\
&\simeq \operatorname{Hom}_B(U', J)
\end{aligned}$$

□

Step 2. Let $X \xrightarrow{\simeq} B$ be a semiprojective resolution over C; see 8.2.

There is then a factorization $X \xrightarrow{\simeq} D \otimes_C X \xrightarrow{\simeq} B$, where the second map is a semiprojective resolution over D, and also a quasi-isomorphism of DG C-modules

$$\operatorname{Hom}_B(U', J) \simeq \operatorname{Hom}_C(X, \operatorname{Hom}_P(U', J))\,.$$

Proof. The DG D-module $D \otimes_C X$ is semiprojective by 8.3(1).

The map $X \xrightarrow{\simeq} B$ induces the vertical arrows in the commutative diagram

$$\begin{tikzcd}
C \otimes_C X \arrow[r, "\simeq", "\gamma\otimes_C X"'] \arrow[d, "\simeq"'] & D \otimes_C X \arrow[d] & \\
C \otimes_C B \arrow[r, "\cong", "\gamma\otimes_C B"'] & D \otimes_C B \arrow[r, equal] & B
\end{tikzcd}$$

The horizontal ones are induced by $\gamma\colon C \to D$; the one at the bottom is bijective, because γ is surjective and C acts on B through γ; the one on the top is a quasi-isomorphism because γ is one and X is semiprojective.

One thus gets a quasi-isomorphism $D \otimes_C X \xrightarrow{\simeq} B$ of DG D-modules. It induces the quasi-isomorphism below, as $\operatorname{Hom}_P(U', J)$ is semiinjective over D, due to 8.3(2):

$$\begin{aligned}\operatorname{Hom}_B(U', J) &\cong \operatorname{Hom}_D(B, \operatorname{Hom}_P(U', J))\\ &\simeq \operatorname{Hom}_D(D \otimes_C X, \operatorname{Hom}_P(U', J))\\ &\cong \operatorname{Hom}_C(X, \operatorname{Hom}_P(U', J))\,.\end{aligned}$$

The first isomorphism reflects the action of $D = B \otimes_P B$ on $\operatorname{Hom}_P(U', J)$. □

Step 3. Let $V \xrightarrow{\simeq} N$ be a semiprojective resolution over B.

Let H denote the complex of P-modules with underlying graded module equal to $\operatorname{Ext}_{P^{\mathsf{e}}}(P, P^{\mathsf{e}})$ and with zero differential.

There is then a quasi-isomorphism of DG C-modules

$$\operatorname{Hom}_P(U', J) \simeq H \otimes_{P^{\mathsf{e}}} (U \otimes_K V)\,.$$

Proof. The semiinjective DG B-module J is semiinjective over P, see 8.3(4). Thus, $U' \simeq \operatorname{Hom}_P(U, W)$ induces the first link in the chain of morphisms of DG D-modules

$$\begin{aligned}\operatorname{Hom}_P(U', J) &\simeq \operatorname{Hom}_P(\operatorname{Hom}_P(U, W), J)\\ &\simeq \operatorname{Hom}_P(W, P) \otimes_P U \otimes_P J\\ &\cong H \otimes_P U \otimes_P J\\ &\simeq H \otimes_P U \otimes_P N\\ &\simeq H \otimes_P U \otimes_P V\\ &\cong H \otimes_{P^{\mathsf{e}}} (U \otimes_K V)\end{aligned}$$

The second one is due to 8.8(1), as W is a bounded complex of finite projective P-modules, U is a complex of finite projective P-modules by 8.3(3), and $U \xrightarrow{\simeq} M$ shows that $\mathrm{H}(U)$ is finite over P and that $\operatorname{pd}_P U$ is finite. The composition

$$\operatorname{Hom}_P(W, P) = \operatorname{Hom}_P(\operatorname{Hom}_P(H, P), P) \cong H\,,$$

where the isomorphism holds because H is finite projective, induces the third map. The next two quasi-isomorphisms are induced by the maps $V \xrightarrow{\simeq} N \xrightarrow{\simeq} J$, because H and U are complexes of projective P-modules. The final isomorphism reflects the action of $P^{\mathsf{e}} = P \otimes_K P$ on $U \otimes_K V$. □

Step 4. There is a quasi-isomorphism of DG C-modules

$$H \otimes_{P^{\mathsf{e}}} (U \otimes_K V) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, U \otimes_K V)\,.$$

Proof. Let $P^{\mathsf{e}} \to E$ be a semiinjective resolution over P^{e}. As H is projective over P, choose a splitting ζ of the surjection of graded P-modules

$$\mathrm{Z}\operatorname{Hom}_{P^{\mathsf{e}}}(P, E) \to \mathrm{H}(\operatorname{Hom}_{P^{\mathsf{e}}}(P, E)) \cong H\,.$$

The first quasi-isomorphism below is ζ followed by $\mathrm{Z}\operatorname{Hom}_{P^{\mathsf{e}}}(P, E) \subseteq \operatorname{Hom}_{P^{\mathsf{e}}}(P, E)$:

$$H \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(P, E) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, E) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}})\,;$$

the remaining ones are standard. They induce a quasi-isomorphism

$$H \otimes_{P^{\mathsf{e}}} (U \otimes_K V) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}}) \otimes_{P^{\mathsf{e}}} (U \otimes_K V)$$

because the complex of P^{e}-modules $U \otimes_K V$ is semiprojective.

The ring P is flat over K, and both M and N have finite flat dimension over P, so $U \otimes_K V$ has finite flat dimension over P^{e}. Thus, Lemma 8.8(2) yields

$$\operatorname{Hom}_{P^{\mathsf{e}}}(A, P^{\mathsf{e}}) \otimes_{P^{\mathsf{e}}} (U \otimes_K V) \simeq \operatorname{Hom}_{P^{\mathsf{e}}}(A, U \otimes_K V)\,. \tag{3.2.1}$$

The last two quasi-isomorphisms above establish the desired assertion. □

Remark 3.3. The quasi-isomorphism (3.2.1) is induced by a canonical morphism ϑ, defined in Lemma 8.8. When $\operatorname{pd}_{P^{\mathsf{e}}} P$ is finite this map is a quasi-isomorphism even when the flat dimension of N over P is not assumed to be finite.

Indeed, in this case the resolution A can be chosen so as to satisfy $A_i = 0$ for $i > \operatorname{pd}_{P^{\mathsf{e}}} P$, in addition to its other properties; see 8.5.

Step 5. For $\bar{U} = S \otimes_B U$ and $\bar{V} = S \otimes_B V$ there are canonical factorizations, where the second maps are semiprojective resolutions over S:

$$U \xrightarrow{\simeq} \bar{U} \xrightarrow{\simeq} M \quad \text{and} \quad V \xrightarrow{\simeq} \bar{V} \xrightarrow{\simeq} N\,,$$

Let $\bar{U} \otimes_K \bar{V} \to E$ be a semiinjective resolution over S^{e}.

There is then a quasi-isomorphism of DG B^{e}-modules

$$\operatorname{Hom}_C(X, \operatorname{Hom}_{P^{\mathsf{e}}}(A, U \otimes_K V)) \simeq \operatorname{Hom}_{S^{\mathsf{e}}}(S, E)\,.$$

Proof. One sees that the induced maps $\bar{U} \to M$ and $\bar{V} \to N$ are semiprojective resolutions over S by arguing as in the proof of the corresponding statement in Step 2. For similar reasons, $S^{\mathsf{e}} \otimes_{B^{\mathsf{e}}} X \to S$ is a semiprojective resolution over S^{e}.

The isomorphisms below come from adjunction formulas, see [16, Ch. VI, §8]:

$$\begin{aligned}
\operatorname{Hom}_C(X, \operatorname{Hom}_{P^{\mathsf{e}}}(A, U \otimes_K V)) &\cong \operatorname{Hom}_{B^{\mathsf{e}}}(X \otimes_A A, U \otimes_K V)\\
&\cong \operatorname{Hom}_{B^{\mathsf{e}}}(X, U \otimes_K V)\\
&\simeq \operatorname{Hom}_{B^{\mathsf{e}}}(X, \bar{U} \otimes_K \bar{V})\\
&\simeq \operatorname{Hom}_{B^{\mathsf{e}}}(X, E)\\
&\cong \operatorname{Hom}_{S^{\mathsf{e}}}(S^{\mathsf{e}} \otimes_{B^{\mathsf{e}}} X, E)\\
&\simeq \operatorname{Hom}_{S^{\mathsf{e}}}(S, E)
\end{aligned}$$

The quasi-isomorphisms $U \otimes_K V \simeq \bar{U} \otimes_K \bar{V} \to E$ induce the first two quasi-isomorphisms, in view of the semiprojectivity of X over B^{e}. The last quasi-isomorphism is induced by $S^{\mathsf{e}} \otimes_{B^{\mathsf{e}}} X \simeq S$, in view of the semiinjectivity of E over S^{e}. □

Step 6. There is a quasi-isomorphism of DG B^{e}-modules

$$\operatorname{Hom}_{S^{\mathsf{e}}}(S, E) \simeq \operatorname{Hom}_S(\operatorname{Hom}_P(M, I), J')\,.$$

Proof. First we assemble a chain of quasi-isomorphisms of DG B^{e}-modules:

$$\begin{aligned}
\operatorname{Hom}_{S^{\mathsf{e}}}(S, E) &\simeq \operatorname{Hom}_C(X, \operatorname{Hom}_{P^{\mathsf{e}}}(A, U \otimes_K V))\\
&\simeq \operatorname{Hom}_C(X, H \otimes_{P^{\mathsf{e}}} (U \otimes_K V))\\
&\simeq \operatorname{Hom}_C(X, \operatorname{Hom}_P(U', J))\\
&\simeq \operatorname{Hom}_B(U', J)\\
&\simeq \operatorname{Hom}_S(\operatorname{Hom}_P(M, I), J')
\end{aligned}$$

The first one is proved in Step 5. The next two are induced by the quasi-isomorphisms of DG C-modules from Steps 4 and 3, due to the semiprojectivity of the DG C-module X. The last two quasi-isomorphisms are proved in Steps 2 and 1. □

A few additional observations now suffice to finish the proof of the theorem.

Step 5 yields quasi-isomorphisms $M\otimes^{\mathbf{L}}_K N \xleftarrow{\simeq} \bar U\otimes_K \bar V \xrightarrow{\simeq} E$, which show that E is a semiinjective resolution of $M\otimes^{\mathbf{L}}_K N$ over S^{e}, so the complex of S-modules $\operatorname{Hom}_{S^{\mathsf{e}}}(S,E)$ represents $\mathbf{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S,M\otimes^{\mathbf{L}}_K N)$.

On the other hand, $W\xrightarrow{\simeq} I$ is a semiinjective resolution over P, and it was shown in Step 1 that $N\xrightarrow{\simeq} J'$ is a semiinjective resolution over S, so the complex of S-modules $\operatorname{Hom}_S(\operatorname{Hom}_P(M,I),J')$ represents $\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_P(M,W),N)$.

The result of Step 6 now proves that the complexes of S-modules

$$\mathbf{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S,M\otimes^{\mathbf{L}}_K N) \quad\text{and}\quad \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_P(M,W),N)$$

are linked by a chain of quasi-isomorphisms of DG B^{e}-modules, where B^{e} acts on them through the composition of morphisms of DG algebras $B^{\mathsf{e}}\to B\to S$. This action coincides with the action through the composition $B^{\mathsf{e}}\to S^{\mathsf{e}}\to S$. Therefore, Lemma 8.4, applied first to the quasi-isomorphism $B^{\mathsf{e}}\to S^{\mathsf{e}}$, then to the homomorphisms $S\to S^{\mathsf{e}}$ and $S^{\mathsf{e}}\to S$ given by $s\mapsto s\otimes 1$ and $s\otimes s'\mapsto ss'$, shows that the complexes above are linked by a chain on isomorphisms in $\mathsf{D}(S)$. □

4. Smooth factorizations

In this section we specialize the results of Section 3.

Recall that when $\varkappa\colon K\to P$ is an essentially smooth homomorphism of commutative rings the P-module of Kähler differentials $\Omega_{P|K}$ is finite and projective. We say that $\varkappa$ has *relative dimension* d if this projective module has rank d; see 2.5.

The next result parallels Theorem 3.1. Note that every homomorphism (essentially) of finite type can be factored through one that is (essentially) smooth of finite relative dimension: the factorization described in 2.2 has these properties.

Theorem 4.1. *Let K be a noetherian ring, $\sigma\colon K\to S$ homomorphism of rings, and $K\xrightarrow{\varkappa} P\xrightarrow{\pi} S$ an essentially smooth factorization of σ of relative dimension d.*

If M and N are complexes of S-modules, such that M is flat over K and $\mathrm{H}(M)$ *is finite over S, then each such factorization of σ yields a quasi-isomorphism*

$$\mathbf{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S,M\otimes_K N)\simeq \Sigma^{-d}\,\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_P(M,{\textstyle\bigwedge}^d_P\Omega_{P|K}),N)$$

of complexes of S-modules that is natural in both M and N.

In particular, for every $n\in\mathbb{Z}$ one has an isomorphism of S-modules

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S,S^{\mathsf{e}})\cong \operatorname{Ext}^{n-d}_S(\mathbf{R}\operatorname{Hom}_P(S,{\textstyle\bigwedge}^d_P\Omega_{P|K}),S)\,.$$

The theorem is proved later in the section. The main ingredient is an explicit computation of the graded P-module $\operatorname{Ext}_{P^{\mathsf{e}}}(P,P^{\mathsf{e}})$. We proceed to describe two specific homomorphisms that are used in the process.

4.2. For each integer n there is a natural homomorphism of P-modules

$$\lambda_n\colon {\textstyle\bigwedge}^n\Omega_{P|K}\longrightarrow \operatorname{Tor}^{P^{\mathsf{e}}}_n(P,P)\,. \tag{4.2.1}$$

Indeed, for $I=\operatorname{Ker}(\mu\colon P^{\mathsf{e}}\to P)$ one has canonical isomorphisms of P-modules

$$\Omega_{P|K}\cong I/I^2\cong \operatorname{Tor}^{P^{\mathsf{e}}}_1(P,P)\,. \tag{4.2.2}$$

As μ is a homomorphism of commutative rings, $\operatorname{Tor}^{P^{\mathsf{e}}}(P,P)$ has a natural structure of a strictly graded-commutative P-algebra, so the composition of the maps above extends to a homomorphism $\lambda\colon {\textstyle\bigwedge}_P\Omega_{P|K}\to \operatorname{Tor}^{P^{\mathsf{e}}}(P,P)$ of graded P-algebras.

The map λ plays a prominent role in classical characterizations of smoothness. We recall a few of them, for use in proofs in this paper and also for comparison with the characterizations of Gorenstein homomorphisms in Theorem 2.1.

4.3. Let $\varkappa\colon K \to P$ be a flat and essentially of finite type homomorphism of rings and set $I = \operatorname{Ker}(P^{\mathsf{e}} \to P)$. The following conditions are equivalent.

(i) The homomorphism $\varkappa$ is essentially smooth.
(ii) For each $\mathfrak{m} \in \operatorname{Supp}_{P^{\mathsf{e}}}(P)$ the ideal $I_{\mathfrak{m}}$ is generated by a regular sequence.
(iii) The P^{e}-module P has finite projective dimension.
(iv) The P-module $\Omega_{P|K}$ is projective, and the homomorphism λ_n from (4.2.1) is bijective for each $n \in \mathbb{Z}$.

Indeed, the equivalence of (i), (ii), and (iv) is due to Hochschild, Kostant, and Rosenberg when K is a perfect field, and to André [1, Prop. C] in general. The equivalence of (i) and (iii) is due to Rodicio [18, Cor. 2].

In our computation we also use another specific homomorphism.

4.4. For each integer n there is a natural homomorphism of P-modules

$$\tau_n\colon \operatorname{Tor}_n^{P^{\mathsf{e}}}(P,P) \longrightarrow \operatorname{Hom}_P(\operatorname{Ext}^n_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}),P)\,.$$

When X is a projective resolution $X \xrightarrow{\simeq} P$ over P^{e}, it can be described as the map in the top row of the commutative diagram an P-linear homomorphisms

$$\begin{tikzcd}
\mathrm{H}_n(X \otimes_{P^{\mathsf{e}}} P) \arrow[r, "\tau_d"] \arrow[d, "\mathrm{H}_n(\delta)"'] & \operatorname{Hom}_P(\mathrm{H}_n(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}})),P) \arrow[d, equal] \\
\mathrm{H}_n(\operatorname{Hom}_{P^{\mathsf{e}}}(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}}),P)) \arrow[r, "\kappa_n"] & \operatorname{Hom}_{P^{\mathsf{e}}}(\mathrm{H}_n(\operatorname{Hom}_{P^{\mathsf{e}}}(X,P^{\mathsf{e}})),P)
\end{tikzcd}$$

where δ is a canonical morphism of complexes and κ_n is a Künneth map.

Relative dimension controls some homological invariants of smooth maps.

4.5. If $K \to P$ is essentially smooth of relative dimension d, then $I_{\mathfrak{m}}$ is generated by a regular sequence of length d for each $\mathfrak{m} \in \operatorname{Supp}_{P^{\mathsf{e}}}(P)$, and $\operatorname{pd}_{P^{\mathsf{e}}}(P) = d$.

Indeed, in view of 4.3 for the first assertion only the length of the sequence is at issue; it is detected by the isomorphisms $(I/I^2)_{\mathfrak{m}} \cong (\Omega_{P|K})_{\mathfrak{m}} \cong (P\mathfrak{m})^d$ of $P_{\mathfrak{m}}$-modules. Using Koszul complexes to resolve $P_{\mathfrak{m}}$ over $P^{\mathsf{e}}_{\mathfrak{m}}$ one obtains $\operatorname{pd}_{P^{\mathsf{e}}_{\mathfrak{m}}}(P_{\mathfrak{m}}) = d$ for every $\mathfrak{m} \in \operatorname{Supp}_{P^{\mathsf{e}}} P$, and this implies $\operatorname{pd}_{P^{\mathsf{e}}} P = d$.

Theorem 4.6. *If K is a noetherian ring and $\varkappa\colon K \to P$ an essentially smooth homomorphism of rings of relative dimension d, then the canonical maps defined in* 4.2 *and* 4.4 *define an isomorphism of P-modules*

$$\tau_d \circ \lambda_d\colon {\textstyle\bigwedge}^d_P \Omega_{P|K} \xrightarrow{\cong} \operatorname{Hom}_P(\operatorname{Ext}^d_{P^{\mathsf{e}}}(P,P^{\mathsf{e}}),P)\,.$$

Proof. In view of 4.3, we have to show that τ_d is bijective. To this end, note that it can be computed from any projective resolution X by finite projective P^{e}-modules. The map δ in the decomposition of τ_d in 4.4 then is bijective. To prove that so is κ_d, we show that $(\kappa_d)_{\mathfrak{m}\cap P^{\mathsf{e}}}$ is bijective for every prime ideal $\mathfrak{m}$ of P. This map is natural, so it suffices to prove that for $Q = (P^{\mathsf{e}})_{\mathfrak{m}}$ and some free resolution X' of $P_{\mathfrak{m}}$ over Q the following natural map is an isomorphism of $P_{\mathfrak{m}}$-modules:

$$\kappa'_d\colon \mathrm{H}_d(\operatorname{Hom}_Q(\operatorname{Hom}_Q(X',Q),P_{\mathfrak{m}})) \longrightarrow \operatorname{Hom}_Q(\mathrm{H}_d(\operatorname{Hom}_Q(X',Q)),P_{\mathfrak{m}})\,.$$

By 4.5 one can choose X' to be the Koszul complex on a regular sequence of length d. This choice yields an isomorphism of complexes $\operatorname{Hom}_Q(X',Q)\cong\Sigma^{-d}X'$, from where one sees that both the source and the target of κ'_d are isomorphic to $P_{\mathfrak{q}}$. An easy calculation then shows that κ'_d itself is bijective. □

Throughout the rest of the paper we use the fact that finite homological dimensions ascend and descend along smooth homomorphisms.

Lemma 4.7. *Let $\varkappa\colon K\to P$ be a flat homomorphism.*

For every complex M of P-module M the following inequalities hold:

$$\operatorname{fd}_K M\le \operatorname{fd}_P M\le \operatorname{fd}_K M+\operatorname{pd}_{P^{\mathsf{e}}}P\,.$$

When P is noetherian and $\mathrm{H}(M)$ is finite one can replace $\operatorname{fd}_P M$ with $\operatorname{pd}_P M$.

If $\varkappa$ is essentially smooth, then $\operatorname{fd}_P M$ and $\operatorname{fd}_K M$ are finite simultaneously.

Proof. Since P is flat over K, every resolution of M by flat P-modules is also a flat resolution of M over K. This explains the inequality on the left. To prove the one on the right we may assume that $\operatorname{fd}_K M$ is finite, say equal to q. Thus, if $F\to M$ is a semiprojective resolution over P, then $G=\operatorname{Ker}(\partial^F_{q-1})$ is flat a K-module; see 8.7. For each $n\in\mathbb{Z}$ there is a canonical isomorphism of functors of P-modules

$$\operatorname{Tor}^P_n(G,-)\cong\operatorname{Tor}^{P^{\mathsf{e}}}_n(P,G\otimes_K -)$$

so one gets the desired inequalities hold for $\operatorname{fd}_R M$. In case P is noetherian and $\mathrm{H}(M)$ is finite over P one has $\operatorname{fd}_P M=\operatorname{pd}_P M$; see 8.7.

When $\varkappa$ is essentially smooth one has $\operatorname{pd}_{P^{\mathsf{e}}}P<\infty$, see 4.3, so the inequalities above imply that $\operatorname{fd}_P M$ is finite if only if so is $\operatorname{fd}_K M$. □

At this point, it is a mere formality to prove the result at the top of the section.

Proof of Theorem 4.1. Recall that $K\to P\to S$ is an essentially smooth factorization of σ. Lemma 4.7 shows that the complex M satisfies the hypotheses of Theorem 3.1. In view of 4.3 and Remark 3.3, Theorem 3.1 applies to an arbitrary complex N. Finally, Theorem 4.6 supplies the explicit form of the P-module W. □

5. Bigrade of a homomorphism

Invariants provided by Hochschild cohomology reflect the structure of an algebra $\sigma\colon K\to S$ as a *bimodule* over itself. We define its *bigrade* by the formula

$$\operatorname{bigrade}(\sigma)=\inf\{n\in\mathbb{Z}\mid \operatorname{Ext}^n_{S^{\mathsf{e}}}(S,S^{\mathsf{e}})\neq 0\}\,.$$

Applications of this invariant are given in the next two sections. Here we examine its formal properties and compare it to other invariants of the K-algebra S.

We define the *residual transcendence degree* of σ at $\mathfrak{n}\in\operatorname{Spec}S$ to be the number

$$\operatorname{tr}\deg_\sigma k(\mathfrak{n})=\operatorname{tr}\deg_{k(\mathfrak{n}\cap K)}k(\mathfrak{n})\,.$$

Theorem 5.1. *Let K be a noetherian ring, $\sigma\colon K\to S$ a flat homomorphism, and $K\to P\to S$ an essentially smooth factorization of relative dimension d.*

Set $p=\operatorname{pd}_P S$ and let $\mathfrak{q}$ be a minimal prime ideal of S.

The following (in)equalities then hold:

$$0\le d-p\le\operatorname{bigrade}(\sigma)\le\operatorname{bigrade}(\sigma_{\mathfrak{q}})=\operatorname{tr}\deg_\sigma k(\mathfrak{q})\le d\,.$$

One has $\operatorname{bigrade}(\sigma)=d-p$ *if and only if* $\operatorname{Ass}S\cap\operatorname{Supp}_S\operatorname{Ext}^p_P(S,P)\neq\varnothing$.

The statements of the theorem are gleaned from 4.5 and Lemmas 4.7, 5.3, and 5.7. Its hypotheses and notation stay in force for the rest of this section.

5.2. Set $V = \bigwedge_P^d \Omega_{P|k}$. Since V is a projective P-module of rank one, for each $n \in \mathbb{Z}$ and every $\mathfrak{n} \in \operatorname{Spec} S$ there are isomorphisms of $S_{\mathfrak{n}}$-modules

$$\begin{aligned}
\operatorname{Ext}_P^n(S,V)_{\mathfrak{n}} &\cong \operatorname{Ext}_P^n(S,V) \otimes_S S_{\mathfrak{n}} \\
&\cong V \otimes_P \operatorname{Ext}_P^n(S,P) \otimes_S S_{\mathfrak{n}} \\
&\cong V \otimes_P (\operatorname{Ext}_P^n(S,P)_{\mathfrak{n}}) \\
&\cong V_{\mathfrak{n}\cap P} \otimes_{P_{\mathfrak{n}\cap P}} \operatorname{Ext}_P^n(S,P)_{\mathfrak{n}}
\end{aligned}$$

The P-module V is faithfully flat—being projective of rank one—hence

$$\operatorname{Supp}_S \operatorname{Ext}_P^n(S,V) = \operatorname{Supp}_S \operatorname{Ext}_P^n(S,P)\,.$$

Lemma 5.3. *There is an inequality* $\operatorname{bigrade}(\sigma) \geq d - p$.
Equality holds if and only if $\operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}_P^p(S,P) \neq \varnothing$.

Proof. Set $V = \bigwedge_P^d \Omega_{P|k}$, $D = \mathbf{R}\mathrm{Hom}_P(S,V)$ and $C = \mathrm{H}_{-p}(D)$, where $p = \operatorname{pd}_P S$.

From 5.2 one gets $\mathrm{H}_n(D) = 0$ for $n < -p$. This implies the second isomorphism of S-modules below, while Theorem 4.1 gives the first one:

$$\operatorname{Ext}_{S^{\mathsf{e}}}^{n+d}(S,S^{\mathsf{e}}) \cong \operatorname{Ext}_S^n(D,S) \cong \begin{cases} 0 & \text{for } n < -p\,; \\ \operatorname{Hom}_S(C,S) & \text{for } n = -p\,; \end{cases}$$

These isomorphisms yield $\operatorname{bigrade}(\sigma) \geq d - p$ and show that equality is equivalent to $\operatorname{Hom}_S(C,S) \neq 0$. Referring to a standard formula and to 5.2 we obtain

$$\operatorname{Ass}_S \operatorname{Hom}_S(C,S) = \operatorname{Ass} S \cap \operatorname{Supp}_S C = \operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}_P^p(S,P)\,.$$

Thus, $\operatorname{Hom}_S(C,S) \neq 0$ is equivalent to $\operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}_P^p(S,P) \neq \varnothing$. □

Before continuing, we recall another canonical isomorphism.

5.4. Let $\mathfrak{n}$ be a prime ideal in S and set $S^{\mathsf{e}}_{\mathfrak{n}} = S_{\mathfrak{n}} \otimes_{K_{\mathfrak{n}\cap K}} S_{\mathfrak{n}}$.
For each $n \in \mathbb{Z}$ there is an isomorphism of $S_{\mathfrak{n}}$-modules

$$\operatorname{Ext}_{S^{\mathsf{e}}_{\mathfrak{n}}}^n(S_{\mathfrak{n}}, S^{\mathsf{e}}_{\mathfrak{n}}) \cong \operatorname{Ext}_{S^{\mathsf{e}}}^n(S,S^{\mathsf{e}})_{\mathfrak{n}}\,.$$

Indeed, let $\lambda\colon S \to S_{\mathfrak{n}}$ and $\kappa\colon K \to K_{\mathfrak{n}\cap K}$ denote the localization maps. The homomorphism $\lambda \otimes_\kappa \lambda\colon S^{\mathsf{e}} \to S^{\mathsf{e}}_{\mathfrak{n}}$ is flat, and one has an isomorphism $S_{\mathfrak{n}} \cong S^{\mathsf{e}}_{\mathfrak{n}} \otimes_{S^{\mathsf{e}}} S$ of $S^{\mathsf{e}}_{\mathfrak{n}}$-modules, whence the first isomorphism below:

$$\begin{aligned}
\operatorname{Ext}_{S^{\mathsf{e}}_{\mathfrak{n}}}^n(S_{\mathfrak{n}}, S^{\mathsf{e}}_{\mathfrak{n}}) &\cong \operatorname{Ext}_{S^{\mathsf{e}}}^n(S,S^{\mathsf{e}}) \otimes_{S^{\mathsf{e}}} S^{\mathsf{e}}_{\mathfrak{n}} \\
&\cong \operatorname{Ext}_{S^{\mathsf{e}}}^n(S,S^{\mathsf{e}}) \otimes_S S_{\mathfrak{n}}\,.
\end{aligned}$$

For the second one note that S^{e} acts on $\operatorname{Ext}_{S^{\mathsf{e}}}^n(S,S^{\mathsf{e}})$ through S.

Lemma 5.5. *The following equality holds:*

$$\operatorname{bigrade}(\sigma) = \min\{\operatorname{bigrade}(\sigma_{\mathfrak{n}}) \mid \mathfrak{n} \in \operatorname{Spec} S\}\,.$$

Proof. Set $g = \operatorname{bigrade}(\sigma)$. From the isomorphisms in 5.4 one reads off an equality $\operatorname{bigrade}(\sigma_{\mathfrak{n}}) \geq g$, which becomes an equality when $\mathfrak{n}$ is in $\operatorname{Supp}_S \operatorname{Ext}_{S^{\mathsf{e}}}^g(S,S^{\mathsf{e}})$. □

Lemma 5.6. *For each prime ideal* $\mathfrak{m}$ *in* P *one has*

$$\operatorname{tr\,deg}_{\varkappa} k(\mathfrak{m}) \leq d\,.$$

Equality holds when P *is a domain and* $\mathfrak{m} = (0)$.

Proof. Set $k = k(\mathfrak{m} \cap K)$ and $P' = (k \otimes_K P)_{\mathfrak{m}}$.

The composed homomorphism $\varkappa' \colon k \to k \otimes_K P \to P'$ is essentially of finite type. Its fibers are among those of $\varkappa$, so it is essentially smooth, and the canonical isomorphism $\Omega_{P'|k} \cong (k \otimes_K \Omega_{P|K})_{\mathfrak{m}}$ shows that it has relative dimension d.

The surjection $P' \to k(\mathfrak{m})$ induces a surjection $\omega \colon k(\mathfrak{m}) \otimes_{P'} \Omega_{P'|k} \to \Omega_{k(\mathfrak{m})|k}$. This gives the second inequality below; for the first one, see [17, (26.10)]:

$$\operatorname{tr\,deg}_k k(\mathfrak{m}) \leq \operatorname{rank}_{k(\mathfrak{m})} \Omega_{k(\mathfrak{m})|k} \leq d\,.$$

When P is a domain and $\mathfrak{m} = (0)$ one has $P' = k(\mathfrak{m})$. In particular, ω is an isomorphism, and thus the second inequality becomes an equality. The first inequality also does, as the homomorphism $k \to k(\mathfrak{m})$ is essentially smooth. □

Lemma 5.7. *For all prime ideals* $\mathfrak{q} \in \operatorname{Min} S$ *and* $\mathfrak{n} \in \operatorname{Spec} S$ *with* $\mathfrak{q} \subseteq \mathfrak{n}$ *one has*

$$\operatorname{bigrade}(\sigma) \leq \operatorname{bigrade}(\sigma_{\mathfrak{n}}) \leq \operatorname{bigrade}(\sigma_{\mathfrak{q}}) = \operatorname{tr\,deg}_{\sigma} k(\mathfrak{q}) \leq d\,.$$

Proof. Both inequalities on the left come from Lemma 5.5, as one has $\sigma_{\mathfrak{q}} = (\sigma_{\mathfrak{n}})_{\mathfrak{q}S_{\mathfrak{n}}}$.

Set $\mathfrak{p} = \mathfrak{q} \cap K$. The rings $S_{\mathfrak{q}}$ and $K_{\mathfrak{p}}$ are artinian, the first one because the ideal $\mathfrak{q}$ is minimal, the second because $\sigma_{\mathfrak{q}} \colon K_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is a flat local homomorphism.

Set $k = k(\mathfrak{p})$, $l = k(\mathfrak{q})$, and $t = \operatorname{tr\,deg}_k l$. Choose in $S_{\mathfrak{q}}$ elements $y_1, \dots, y_t$ that map to a transcendence basis of l over k. Let $x_1, \dots, x_t$ be indeterminates over $K_{\mathfrak{p}}$ and Q the localization of $K_{\mathfrak{p}}[x_1, \dots, x_t]$ at the prime ideal $\mathfrak{p}K_{\mathfrak{p}}[x_1, \dots, x_t]$; this is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}Q$ and residue field $k' = k(x_1, \dots, x_t)$.

The homomorphism of $K_{\mathfrak{p}}$-algebras $K_{\mathfrak{p}}[x_1, \dots, x_t] \to S_{\mathfrak{q}}$ sending x_i to y_i for $i = 1, \dots, t$ induces a local homomorphism $\varphi \colon Q \to S_{\mathfrak{q}}$. A length count yields

$$\operatorname{length}_Q(S_{\mathfrak{q}}) = \operatorname{length}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \operatorname{length}_{k'}(l) < \infty\,,$$

so φ is finite. Let κ denote the composition $K_{\mathfrak{p}} \to K_{\mathfrak{p}}[x_1, \dots, x_t] \to Q$. It is local, flat, essentially of finite type, and the fiber $Q \otimes_{K_{\mathfrak{p}}} k$ is equal to k'.

One has $\operatorname{Spec} K_{\mathfrak{p}} = \{\mathfrak{p}\}$ and $\sigma_{\mathfrak{q}} = \varphi\kappa$, so this is a smooth factorization of relative dimension t by the discussion above. The finite Q-module $S_{\mathfrak{q}}$ has finite projective dimension by Lemma 4.7, so it is free because Q is artinian. By the same token, one has $\operatorname{Supp}_{S_{\mathfrak{q}}} \operatorname{Ext}^0_Q(S_{\mathfrak{p}}, Q) = \mathfrak{q}S_{\mathfrak{q}} = \operatorname{Ass} S_{\mathfrak{q}}$, so Lemma 5.3 yields $\operatorname{bigrade}(\sigma_{\mathfrak{q}}) = t$.

Finally, set $\mathfrak{m} = \mathfrak{q} \cap P$. As the field extension $k(\mathfrak{m}) \subseteq k(\mathfrak{q})$ is finite, on has $t = \operatorname{tr\,deg}_k k(\mathfrak{m})$. On the other hand, Lemma 5.6 yields $t \leq d$. □

We illustrate the preceding results in a concrete situation. When every associated prime ideal $\mathfrak{q}$ of S satisfies $\dim(S/\mathfrak{q}) = \dim S$ we say that S is *equidimensional*.

Proposition 5.8. *Let* K *be a field and* S *a graded* K*-algebra with* $S_0 = K$*, generated over* S_0 *by finitely many elements of positive degree. One then has*

$$\operatorname{depth} S \leq \operatorname{bigrade}(\sigma) \leq \dim S\,.$$

The first inequality is strict when S *is equidimensional, but not Cohen-Macaulay.*

Proof. Set $d = \dim S$, and let P be the K-subalgebra of S generated by some homogeneous system of parameters. Thus, P is a polynomial ring in d variables and S and $K \to P \to S$ is a smooth factorization of σ of relative dimension d.

Set $p = \operatorname{pd}_S P$. The Auslander-Buchsbaum Equality gives $d - p = \operatorname{depth} S$, so the desired inequalities hold by Theorem 5.1. When S is equidimensional and $\mathfrak{q}$ is in $\operatorname{Ass} S$ one has $\dim(S/\mathfrak{q}) = d = \dim P$. As $S/\mathfrak{q}$ is finite over P and P is a domain, this implies $\mathfrak{q} \cap P = (0)$. When S is not Cohen-Macaulay one has $p > 0$, hence

$$\begin{aligned}\operatorname{Ext}^p_P(S,P)_{\mathfrak{q}} &\cong (\operatorname{Ext}^p_P(S,P) \otimes_P P_{(0)}) \otimes_{P_{(0)}} S_{\mathfrak{q}} \\ &\cong \operatorname{Ext}^p_{P_{(0)}}(S_{(0)}, P_{(0)}) \otimes_{P_{(0)}} S_{\mathfrak{q}} \\ &= 0\end{aligned}$$

From Theorem 5.1 we now conclude that $\operatorname{bigrade}(\sigma) > d - p$ holds. □

Next we show that the second inequality in the proposition can be strict as well.

Example 5.9. When K is a field of characteristic zero the subring

$$S = K[x^3, x^2y, x^2z, xy^2, xz^2, y^3, y^2z, yz^2, z^3]$$

of a polynomial ring $K[x,y,z]$ and the inclusion map $\sigma\colon K \to S$ satisfy

$$\operatorname{bigrade}(\sigma) = 2 < 3 = \dim S\,.$$

Indeed, for the equality on the right note that the field of fractions of S is equal to $K(x,y,z)$. The one on the left results from the isomorphisms

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \cong \begin{cases} 0 & \text{for } n \leq 1\,; \\ K & \text{for } n = 2\,. \end{cases}$$

For $K = \mathbb{Q}$ they are established through a computation with *Macaulay 2*. The general case follows from here and the formal property below.

5.10. For every homomorphism of rings $K \to K'$ we identify K' and $K \otimes_K K'$ via the canonical isomorphism, set $S' = S \otimes_K K'$, and note that $\sigma \otimes_K K'\colon K' \to S'$ is (essentially) of finite type and flat, along with σ. Also, set $S'^{\mathsf{e}} = S' \otimes_{K'} S'$.

When $K \to K'$ is flat so is $S^{\mathsf{e}} \to S'^{\mathsf{e}}$, due to the canonical isomorphism of K'-algebras $S'^{\mathsf{e}} \cong S^{\mathsf{e}} \otimes_K K'$. Thus, for each $n \in \mathbb{Z}$ one gets isomorphisms

$$\operatorname{Ext}^n_{S'^{\mathsf{e}}}(S', S'^{\mathsf{e}}) \cong \operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S'^{\mathsf{e}}) \cong \operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \otimes_S S'\,.$$

As a consequence, for every flat homomorphism $K \to K'$ one obtains

$$\operatorname{bigrade}(\sigma) \leq \operatorname{bigrade}(\sigma \otimes_K K')\,;$$

equality holds when K' is faithfully flat over K.

6. Cohen-Macaulay homomorphisms

Let $\sigma\colon K \to S$ be a flat homomorphism of noetherian rings. It is said to be *Cohen-Macaulay at a prime ideal* $\mathfrak{n}$ of S if the local ring $S_{\mathfrak{n}}/(\mathfrak{n} \cap K)S$ is Cohen-Macaulay; it is *Cohen-Macaulay* if it has this property at each $\mathfrak{n} \in \operatorname{Spec} S$.

Theorem 6.1. *Let K be a noetherian ring and $\sigma\colon K \to S$ a flat Cohen-Macaulay homomorphism essentially of finite type.*

If $\operatorname{Spec} S$ *is connected, then for all* $\mathfrak{q} \in \operatorname{Min} S$ *and* $\mathfrak{n} \in \operatorname{Spec} S$ *one has*

$$\operatorname{bigrade}(\sigma) = \operatorname{bigrade}(\sigma_{\mathfrak{n}}) = \operatorname{tr\,deg}_{\sigma} k(\mathfrak{q})\,.$$

Theorem 6.2. *Let K be a noetherian ring, $\sigma\colon K \to S$ a flat Cohen-Macaulay homomorphism with* $\operatorname{Spec} S$ *connected, and $K \to P \to S$ an essentially smooth surjective factorization of σ of relative dimension d; set $p = pd_P S$.*

The S-module $C = \operatorname{Ext}^p_P(S, \bigwedge^d_P \Omega_{P\,|\,K})$ then has the following properties.

(1) *There is an equality* $\operatorname{Ass}_S C = \operatorname{Ass} S$.

(2) *For each* $n \in \mathbb{Z}$ *there is an isomorphism of* S*-modules*

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \cong \operatorname{Ext}^{n-g}_S(C, S) \quad \textit{where} \quad g = \operatorname{bigrade}(\sigma)\,.$$

(3) *When* K *is Gorenstein* $C_{\mathfrak{n}}$ *is a canonical module for* $S_{\mathfrak{n}}$ *for each* $\mathfrak{n} \in \operatorname{Spec} S$.

Connectedness is assumed only for ease of exposition; see 6.5. The theorems are proved concurrently, after some characterizations of Cohen-Macaulay maps.

6.3. Let K be a noetherian ring, $\sigma\colon K \to S$ a flat homomorphism, and $K \to P \to S$ an essentially smooth surjective factorization of σ.

6.3.1. For each $\mathfrak{n} \in \operatorname{Spec} S$ the following inequalities hold:

$$\operatorname{grade}_P S \leq \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \leq \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \leq \operatorname{pd}_P S \leq d\,.$$

The first three are standard, the last one comes from Lemma 4.7.

6.3.2. The homomorphism σ is Cohen-Macaulay if and only if for every $\mathfrak{n} \in \operatorname{Spec} S$ one has $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$; this follows from [7, (3.5) and (3.7)].

Theorem 6.4. *Let* K *be a noetherian ring,* $\sigma\colon K \to S$ *a flat homomorphism, and* $K \to P \to S$ *an essentially smooth surjective factorization of relative dimension* d.

When $\operatorname{Spec} S$ *is connected the following conditions are equivalent.*

(i) σ *is Cohen-Macaulay.*
(ii) $\operatorname{grade}_P S = \operatorname{pd}_P S$.
(iii) $\operatorname{Ext}^n_P(S, P) = 0$ *for* $\operatorname{grade}_P S < n \leq d$.

When S *and* P *are integral domains one has*

$$\operatorname{grade}_P S = \operatorname{tr\,deg}_K P - \operatorname{tr\,deg}_K S \quad \textit{and} \quad d = \operatorname{tr\,deg}_K P\,.$$

Proof. (i) $\implies$ (iii). Let $\mathfrak{n}$ and $\mathfrak{n}'$ be prime ideal in S. We first prove that one has

$$\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{grade}_{P_{\mathfrak{n}'\cap P}} S_{\mathfrak{n}'} = \operatorname{pd}_{P_{\mathfrak{n}'\cap P}} S_{\mathfrak{n}'}\,.$$

For the first and last equalities it suffices to remark that $\sigma_{\mathfrak{n}}$ and $\sigma_{\mathfrak{n}'}$ are Cohen-Macaulay along with σ, then refer to 6.3.2. When $\mathfrak{n}$ is contained in $\mathfrak{n}'$ the equality in the middle is obtained by applying the chain of inequalities in 6.3.1 to $\sigma_{\mathfrak{n}'}$. Since $\operatorname{Spec} S$ is connected, when $\mathfrak{n}$ and $\mathfrak{n}'$ are arbitrary one can find in $\operatorname{Spec} S$ a path

$$\mathfrak{n} = \mathfrak{n}_0 \supseteq \mathfrak{n}_1 \subseteq \mathfrak{n}_2 \subseteq \cdots \supseteq \mathfrak{n}_{j-1} \subseteq \mathfrak{n}_j = \mathfrak{n}'$$

The already treated case of embedded prime ideals shows that the invariants that we are tracking stay constant on each segment of such a path.

When $\mathfrak{n}$ ranges over $\operatorname{Spec} S$ the infimum of $\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ equals $\operatorname{grade}_P S$, and the supremum of $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ equals $\operatorname{pd}_P S$, so these two numbers are equal.

(ii) $\implies$ (i). By 6.3.1 the hypothesis implies that for every $\mathfrak{n} \in \operatorname{Spec} S$ one has $\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$. This means that σ is Cohen-Macaulay, see 6.3.2.

(ii) $\implies$ (iii). This is evident.

(iii) $\implies$ (ii). The definition of grade and the hypothesis imply that for $n \in [0, d]$ one has $\operatorname{Ext}^n_P(S, P) \neq 0$ only when $n = \operatorname{grade}_P S$. On the other hand, Lemma 4.7 yields $\operatorname{pd}_P S \leq d$; since P is noetherian and S is a finite P-module this implies $\operatorname{Ext}^n_P(S, P) \neq 0$ for $n = \operatorname{pd}_P S$. One concludes that $\operatorname{pd}_P S = \operatorname{grade}_P S$ holds.

Assume now that S and P are integral domains, and set $\mathfrak{m} = \operatorname{Ker}(P \to S)$.

Since $S_{\mathfrak{n}}$ has finite projective dimension over $P_{\mathfrak{n}\cap P}$, see Lemma 4.7, from [7, (2.5)] one gets $\dim P_{\mathfrak{m}} = \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$. Thus, one obtains

$$\operatorname{grade}_P S = \inf\{\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \,|\, \mathfrak{n} \in \operatorname{Spec} S\} = \dim P_{\mathfrak{m}}\,.$$

Set $S' = S_{(0)}$ and $K' = K_{(0)\cap K}$. As S' is the residue field of $P_{\mathfrak{m}}$, and $K' \to S'$ is a flat local homomorphism, one sees that K' is a field. The local domain $P_{\mathfrak{m}}$ is the localization of some finitely generated K'-algebra P' at a prime ideal $\mathfrak{m}'$, so one has

$$\dim P_{\mathfrak{m}} = \operatorname{height}(\mathfrak{m}') = \dim P' - \dim(P'/\mathfrak{m}') = \operatorname{tr}\deg_K P - \operatorname{tr}\deg_K S\,.$$

To finish the proof, we note that $d = \operatorname{tr}\deg_K P$ holds by Lemma 5.6. $\square$

Proof of Theorems 6.1 *and* 6.2. Set $V = \bigwedge_P^d \Omega_{P|K}$.

The preceding lemma yields $\operatorname{grade}_P S = \operatorname{pd}_P S$, so $\operatorname{Ext}_P^n(S,V) = 0$ for $n \neq p$ holds by 5.2. Thus, from a resolution $F \xrightarrow{\simeq} S$ with each F_i finite projective over P and $F_i = 0$ for $i \notin [0,p]$ one gets quasi-isomorphism of complexes

$$\mathbf{R}\operatorname{Hom}_P(S,V) \simeq \operatorname{Hom}_P(F,V) \simeq \Sigma^{-p} C \tag{6.4.1}$$

of P-modules. Each $\operatorname{Hom}_P(F_i,V)$ is finite projective, and one has isomorphisms

$$\begin{aligned}\operatorname{Hom}_P(\operatorname{Hom}_P(F,V),V) &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P)\otimes_P V, V)\\ &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P),\operatorname{Hom}_P(V,V))\\ &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P),P)\\ &\cong F\end{aligned}$$

of complexes. These computations localize, so for each $\mathfrak{n} \in \operatorname{Spec} S$ one gets

$$\operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}\,.$$

We are ready to prove Theorem 6.2(1). A prime ideal $\mathfrak{n}$ in $\operatorname{Spec} S$ is associated to C if and only if the $S_{\mathfrak{n}}$-module $C_{\mathfrak{n}}$ has depth zero. Invoking first the finiteness of $S_{\mathfrak{n}}$ as a $P_{\mathfrak{n}\cap P}$-module, then the Auslander-Buchsbaum Equality, one gets

$$\operatorname{depth}_{S_{\mathfrak{n}}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} P_{\mathfrak{n}\cap P} - \operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}}\,.$$

Thus, $\mathfrak{n}$ is in $\operatorname{Ass}_S C$ if and only if one has $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} P_{\mathfrak{n}\cap P}$. Similarly, one sees that $\mathfrak{n}$ is in $\operatorname{Ass} S$ if and only if $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$. As the projective dimensions are equal, we get $\operatorname{Ass}_S C = \operatorname{Ass} S$, as desired.

From (6.4.1) and Theorem 4.1 for each $n \in \mathbb{Z}$ one gets

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \cong \operatorname{Ext}^{n-d+p}_S(C,S)\,.$$

We have just proved $\operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}_P^p(S,P) \neq \varnothing$, so Theorem 5.1 implies

$$\operatorname{bigrade}(\sigma) = d - p\,. \tag{6.4.2}$$

These formulas establish Theorem 6.2(2).

Assume that K is Gorenstein. The rings P and S then are Gorenstein and Cohen-Macaulay, respectively, because they are flat over K with fibers of the corresponding type. For each $\mathfrak{n} \in \operatorname{Spec} S$ one has $\operatorname{pd}_{P_{\mathfrak{m}}} S_{\mathfrak{n}} = p$ by Lemma 6.4, and also $C_{\mathfrak{n}} \cong \operatorname{Ext}^p_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P})$ because the P-module C is projective of rank one; see 5.2. Thus, $C_{\mathfrak{n}}$ is a canonical module for $S_{\mathfrak{n}}$; see [10, (3.3.7)]. This proves Theorem 6.2(3).

To prove Theorem 6.1, pick $\mathfrak{n}$ in $\operatorname{Spec} S$ and form the map $\sigma_{\mathfrak{n}}\colon K_{\mathfrak{n}\cap K} \to S_{\mathfrak{n}}$. The induced maps $K_{\mathfrak{n}\cap K} \to P_{\mathfrak{n}\cap P} \to S_{\mathfrak{n}}$ provide a surjective smooth decomposition of relative dimension d. Lemma 6.4 shows that $\sigma_{\mathfrak{n}}$ is Cohen-Macaulay, and $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = p$. Formula (6.4.2) applied to $\sigma_{\mathfrak{n}}$ yields $\operatorname{bigrade}(\sigma) = \operatorname{bigrade}(\sigma_{\mathfrak{n}})$. This proves the first equality. Theorem 5.1 gives the second one. $\square$

6.5. Let $\tau\colon K \to T$ be a flat homomorphism essentially of finite type, set $R = S \times T$ and let $\delta\colon K \to K \times K$ be the diagonal map. For each $n \in \mathbb{Z}$ one has an isomorphism

$$\operatorname{Ext}^n_{R^{\mathsf{e}}}(R, R^{\mathsf{e}}) \cong \operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \oplus \operatorname{Ext}^n_{T^{\mathsf{e}}}(T, T^{\mathsf{e}})\,.$$

of R-modules. In particular, the following equality holds:

$$\operatorname{bigrade}((\sigma \times \tau)\delta) = \min\{\operatorname{bigrade}(\sigma)\,,\ \operatorname{bigrade}(\tau)\}\,.$$

Indeed, set $R = S \times T$ and let e and f denote the orthogonal idempotents $(1, 0)$ and $(0, 1)$ of R. Any R-module N, viewed as a R^{e}-module through the multiplication map, satisfies $(e \otimes f)N = 0 = (f \otimes e)N$, and hence decomposes as $(e \otimes e)N \oplus (f \otimes f)N$. The equalities $(e \otimes e)R = S$ and $(e \otimes e)R^{\mathsf{e}} = S^{\mathsf{e}}$ yield

$$\begin{aligned} S \operatorname{Ext}^n_{R^{\mathsf{e}}}(R, R^{\mathsf{e}}) &= (e \otimes e) \operatorname{Ext}^n_{R^{\mathsf{e}}}(R, R^{\mathsf{e}}) \\ &\cong \operatorname{Ext}^n_{R^{\mathsf{e}}}((e \otimes e)R, (e \otimes e)R^{\mathsf{e}}) \\ &= \operatorname{Ext}^n_{R^{\mathsf{e}}}(S, S^{\mathsf{e}}) \\ &\cong \operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})\,. \end{aligned}$$

Similar isomorphisms involving $(f \otimes f)$ finish the proof.

7. Gorenstein homomorphisms

In Section 2 we proved that an algebra S essentially of finite type is Gorenstein if and only if the graded S-module $\operatorname{Ext}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})$ is projective of rank one. We refine this result to a kind of structure theorem for Gorenstein homomorphisms.

Theorem 7.1. *When K is a noetherian ring and $\sigma\colon K \to S$ a Gorenstein flat homomorphism essentially of finite type there exist uniquely defined rings $S_1, \dots, S_q$ and distinct non-negative integers $n_1, \cdots, n_q$ with the following properties:*

(a) $S = S_1 \times \cdots \times S_q$.
(b) $\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \neq 0$ *if and only if* $n \in \{n_1, \dots, n_q\}$.
(c) $\operatorname{Ext}^{n_i}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) = S_i \operatorname{Ext}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})$, *and this S_i-module is projective of rank one.*
(d) $\operatorname{tr}\deg_\sigma k(\mathfrak{q}) = n_i$ *holds for every* $\mathfrak{q} \in \operatorname{Min} S_i$.

Proof. For each integer $n \geq 0$ form the S-module $E^n = \operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})$.

By Theorem 2.1, the module $E = \bigoplus_{n=0}^{\infty} E^n$ is finite, projective, and has rank one. Thus, each E^n is finite projective, there are only finitely many integers $n_1, \dots, n_q$ with $E^{n_j} \neq 0$, and one has $E = \bigoplus_{i=1}^{q} E^{n_i}$. For each $\mathfrak{n} \in \operatorname{Spec} S$ one has $E_{\mathfrak{n}} = \bigoplus_{i=1}^{q} E^{n_i}_{\mathfrak{n}}$. The $S_{\mathfrak{n}}$-module $E_{\mathfrak{n}}$ is isomorphic to $S_{\mathfrak{n}}$, so it is indecomposable, and hence $E^{n_i}_{\mathfrak{n}} \neq 0$ holds for a unique value of i. Thus, one obtains a disjoint union

$$\operatorname{Spec} S = \operatorname{Supp}_S E = \bigsqcup_{i=1}^{q} \operatorname{Supp}_S E^{n_i}\,.$$

Each set $\operatorname{Supp}_S E^{n_i}$ is closed in $\operatorname{Spec} S$, because E^{n_i} is finite, so this equality yields a decomposition $S = \prod_{i=1}^{q} S_i$, such that $E^{n_i} = S_i E$. Properties (a) through (c) have now been proved. Property (d) comes from Theorem 6.1. □

The criterion below contains Theorem 1 from the introduction. We say that S is *generically Gorenstein* if for each $\mathfrak{q} \in \operatorname{Min} S$ the ring $S_{\mathfrak{q}}$ is Gorenstein.

Theorem 7.2. *Let S be a Cohen-Macaulay ring with connected spectrum, K a Gorenstein ring, $K \to S$ a flat homomorphism essentially of finite type, $\mathfrak{q}$ a minimal prime ideal of S, and set $t = \operatorname{tr}\deg_\sigma k(\mathfrak{q})$.*

The ring S is Gorenstein if and only if it is generically Gorenstein and one has

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) = 0 \quad \textit{for} \quad t < n \le t + \dim S\,.$$

Proof. If S is Gorenstein it is generically Gorenstein, and σ is Gorenstein because it is flat with Gorenstein target ring; see [17, (23.4)]; now apply Theorem 7.1.

For the converse, note that the homomorphism σ is Cohen-Macaulay because it is flat and its target is a Cohen-Macaulay ring; see [17, (23.3), Cor.]. Theorem 6.2(3) then yields a finite S-module C, such that for each $\mathfrak{n} \in \operatorname{Spec} S$ the localization $C_{\mathfrak{n}}$ is a canonical module for the Cohen-Macaulay local ring $S_{\mathfrak{n}}$. Theorems 6.2(2) and 6.1 translate our hypothesis into $\operatorname{Ext}^n_{S_{\mathfrak{n}}}(C_{\mathfrak{n}}, S_{\mathfrak{n}}) = 0$ holds for $1 \le n \le \dim S$.

Thus, it suffices to apply the following result of [3, (2.1)] or [15, (2.2)]:

If S is a generically Gorenstein Cohen-Macaulay local ring with canonical module C, and $\operatorname{Ext}^n_S(C, S) = 0$ holds for $1 \le n \le \dim S$, then S is Gorenstein. □

The results above strongly suggest that for an algebra essentially of finite type the Gorenstein property may be read off the behavior of its Hochschild cohomology. We highlight one of several natural directions in which one might look for extensions:

Conjecture. When K is a Gorenstein ring and S is a finite K-algebra essentially of finite type with connected spectrum, then $\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) = 0$ for all $n > \operatorname{grade}_{S^{\mathsf{e}}} S$ implies that the ring S is Gorenstein.

When the ring S is Cohen-Macaulay it coincides with an earlier conjecture, see [3, Intr.]; this was shown and exploited in the proof of Theorem 7.2. Specializing still further, one encounters Tachikawa's conjecture [19, p. 115]; see Remark 3.2.

To finish, we compare the conditions used in this paper to characterize the Gorenstein property with results drawn from two different pools: Global descriptions of smoothness in terms of similar invariants, see [9], and descriptions of local regular rings and Gorenstein rings in terms of their residue fields, see [17].

7.3. Let $\sigma\colon K \to S$ be a flat homomorphism, essentially of finite type.

Let R be a local ring with residue field k.

Theorem 2.1 can be compared to the following statements: σ is essentially smooth if and only if the S^{e}-module S has finite projective dimension; R is regular if and only if the R-module k has finite projective dimension; R is Gorenstein if and only if the R-module R has finite injective dimension.

Similarly, Theorem 7.2 can be compared to the following statements: σ is smooth if $\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S) = 0$ holds for $\dim S + 2$ consecutive positive values of n, only if $\operatorname{Ext}_{S^{\mathsf{e}}}(S, S) = 0$ all $n \gg 0$; R is regular if $\operatorname{Ext}^n_R(k, k) = 0$ holds for some positive n, only if it does for all $n \gg 0$; R is Gorenstein if $\operatorname{Ext}^n_R(k, R) = 0$ holds for some $n > \operatorname{grade}_R k$, only if it holds for all $n > \operatorname{grade}_R k$.

8. Resolutions

In this section we discuss some homological notions used in the paper.

The main results in this article concern rings and modules over them, but several hinge on Theorem 3.1, whose proof uses DG (short for: differential graded) algebras and DG modules. The material below is tailored to the needs of that argument.

Rings are viewed as DG algebras concentrated in degree zero. With this understanding, DG modules over a ring are nothing but complexes over it, and modules are complexes concentrated in degree zero.

Let A be a DG algebra, that we assume $\mathbb{N}$-graded and graded-commutative; that is, $ab = (-1)^{|a||b|}ba$ holds for all $a, b \in A$, where $|a|$ denotes the degree of a.

The graded-commutativity of A implies that each DG left A-module M has a natural structure of a DG right A-module, given by:

$$m \cdot a = (-1)^{|a||m|}am \quad \text{for } a \in A \text{ and } m \in M.$$

Moreover, given DG modules M and N over A, the module of homomorphisms $\operatorname{Hom}_A(M, N)$ and the tensor product $M \otimes_A N$ are both DG modules over A.

8.1. Let A and B be DG algebras over a commutative ring K. The tensor product $A \otimes_K B$ is again the graded-commutative algebra with product defined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|}(aa' \otimes bb')\,.$$

Let M and N be DG modules over A and B, respectively, and let L be a DG module L over $B \otimes_K A$. By [16, Chapter VI, (8.3),(8.7)], there are natural isomorphisms:

$$\operatorname{Hom}_{B\otimes_K A}(L, \operatorname{Hom}_K(M, N)) \cong \operatorname{Hom}_B(L \otimes_A M, N)$$
$$L \otimes_{A\otimes_K B} (M \otimes_K N) \cong (L \otimes_A M) \otimes_B N$$

We write A^{e} for the DG algebra $A \otimes_K A$. Setting with $B = A = L$ in the isomorphism above one obtains isomorphisms of DG modules over A:

$$\operatorname{Hom}_{A^{\mathsf{e}}}(A, \operatorname{Hom}_K(M, N)) \cong \operatorname{Hom}_A(M, N)$$
$$(M \otimes_K N) \otimes_{A^{\mathsf{e}}} A \cong M \otimes_A N$$

We say that a DG module P over A is *semiprojective* if the functor $\operatorname{Hom}_A(P, -)$ preserves surjections and quasi-isomorphisms, and is *semiflat* if the functor $P \otimes_A -$ preserves injections and quasi-isomorphisms. Every semiprojective DG module is semiflat. A DG module I is *semiinjective* if $\operatorname{Hom}_A(-, I)$ transforms injections into surjections and preserves quasi-isomorphisms.

8.2. Each DG module M over A admits a semiprojective resolution and a semiinjective resolution; that is to say, a surjective quasi-isomorphism $P \to M$ with P semiprojective, and an injective quasi-isomorphism $M \to I$ with I semiinjective. It is defined uniquely up to homotopy equivalences inducing the identity on M.

The following properties track semiprojectivity and semiinjectivity under change of DG algebras. They readily follow from standard adjunction formulas.

8.3. Let $\alpha\colon A \to B$ be a morphism of DG K-algebras, and let X and Y be DG modules over A and B, respectively. The following statements hold:

1. If X is semiprojective, then so is the DG B-module $B \otimes_A X$.
2. If X is semiinjective, then so is the DG B-module $\operatorname{Hom}_A(B, X)$.
3. If B is semiprojective over A and Y is semiprojective over B, then Y is semiprojective over A.
4. If B is semiprojective over A and Y is semiinjective over B, then Y is semiinjective over A.
5. If X is semiprojective and Y is semiinjective, then $\operatorname{Hom}_K(X, Y)$ is semiinjective as a DG module over $A \otimes_K B$.

Lemma 8.4. *Let $\alpha\colon A \to B$ be a morphism of DG algebras, and let Y and Y' be DG B-modules that are quasi-isomorphic when viewed as DG A- modules.*

If α is a quasi-isomorphism, or if there exists a morphism $\beta\colon B \to A$, such that $\alpha\beta = \mathrm{id}^B$, then Y and Y' are quasi-isomorphic as DG B-modules.

Proof. By hypothesis, one has A-linear quasi-isomorphisms $Y \xleftarrow{v} U \xrightarrow{v'} Y'$.

When α is a quasi-isomorphism choose U semiprojective over A, using 8.2. With horizontal arrows defined to be $b \otimes u \mapsto bv(u)$ and $b \otimes u \mapsto bv'(u)$ the diagram

$$\begin{tikzcd}
& U \arrow[ddl, "v"', "\simeq"] \arrow[d, equal] \arrow[ddr, "v'", "\simeq"'] & \\
& A \otimes_A U \arrow[d, "\alpha\otimes_A U"', "\simeq"] & \\
Y & B \otimes_A U \arrow[l] \arrow[r] & Y'
\end{tikzcd}$$

commutes. All the maps above are morphisms of DG B-modules, and $\alpha \otimes_A U$ is a quasi-isomorphism because α is one and U is semiprojective.

When α has a right inverse β, note that the A-linear quasi-isomorphisms v and v' are also B-linear, and that the DG B-module structures on Y and Y' induced via β are identical with their original structures over B. ☐

We present some result on resolutions with DG structure.

8.5. Let $\varphi\colon R' \to Q$ be a homomorphism of rings.

A *DG algebra resolution* of Q over R' is a factorization $R' \to A \xrightarrow{\simeq} Q$ of φ as a composition of morphisms of DG algebras, such that the R'-module A_i is projective for each $i \in \mathbb{Z}$ and equal to 0 for $i < 0$. Such a resolution always exists; when R' is noetherian and φ is finite there is one where each A_i is a finite R'-module.

8.6. Let R be a noetherian ring and B a DG algebra with $B_i = 0$ for $i < 0$ and B_i a finite projective R-module for each $i \in \mathbb{Z}$. Let V be a DG module over B.

If $\mathrm{H}_n(V)$ is finite for each n and one has $\mathrm{H}_{\leqslant i}(V) = 0$ for some $i \in \mathbb{Z}$, then there is a quasi-isomorphism $X \xrightarrow{\simeq} V$ of DG modules over B, with X_n a finite R-module for each n, and $X_n = 0$ for all $n \notin [i, \mathrm{pd}_R V]$.

Next we recall a few notions specific to rings.

8.7. Let Q be a commutative ring.

A complex P of projective modules, with $P_n = 0$ for $n \ll 0$ is semiprojective, while a complex I of injective modules, with $I_n = 0$ for $n \gg 0$ is semiinjective.

The *projective dimension* of a M complex of Q-modules is the number

$$\mathrm{pd}_R M = \inf_n \left\{ n \,\middle|\, \begin{matrix} n \geq \sup \mathrm{H}(M) \text{ and there exists a semiprojective} \\ \text{resolution } P \xrightarrow{\simeq} M \text{ with } \mathrm{Coker}(\partial_{n+1}) \text{ projective} \end{matrix} \right\} .$$

Replacing 'semiprojective' and 'projective' with 'semiflat' and 'flat' respectively, gives the definition of the *flat dimension* of M, denoted $\mathrm{fd}_Q M$. Evidently, one has an inequality $\mathrm{fd}_Q M \leq \mathrm{pd}_Q M$; equality holds if Q is noetherian and $\mathrm{H}(M)$ is finite.

Lemma 8.8. *Let L, M, and N be complexes over a commutative ring Q.*

There are then uniquely defined natural morphisms of complexes of Q- modules

$$\begin{aligned}\theta\colon \operatorname{Hom}_Q(L,Q)\otimes_Q M\otimes_Q N &\longrightarrow \operatorname{Hom}_Q(\operatorname{Hom}_Q(M,L),N)\,;\\ \vartheta\colon \operatorname{Hom}_Q(L,Q)\otimes_Q N &\longrightarrow \operatorname{Hom}_Q(L,N)\,,\end{aligned}$$

satisfying the following conditions:

$$\begin{aligned}\big(\theta(\lambda\otimes m\otimes n)\big)(\delta) &= (-1)^{(|m|+|n|)|\delta|}\lambda\delta(m)\,n\,;\\ \big(\vartheta(\lambda\otimes n)\big)(l) &= (-1)^{|n||l|}\lambda(l)\,n\,.\end{aligned}$$

When L is a complex of finite projective modules the following hold.

1. *The map θ is a quasi-isomorphism if $\operatorname{pd}_Q L$ is finite and M is a homologically finite complex with $\operatorname{pd}_Q M$ finite.*
2. *The map ϑ is a quasi-isomorphism if L is bounded below and N is a bounded below complex with $\operatorname{fd}_Q N$ finite.*

Proof. Indeed, direct calculations establish that the formulas above define morphisms of complexes of Q-modules, which are evidently natural.

(1) In this case there are homotopy equivalences $L'\to L$ and $M\to M'$, with L' and M' bounded complexes of finite projective Q-modules. These maps induce the vertical arrows in the commutative diagram of morphisms of complexes

$$\begin{tikzcd}
\operatorname{Hom}_Q(L,Q)\otimes_Q M\otimes_Q N \arrow[r, "\theta"] \arrow[d, "\simeq"'] & \operatorname{Hom}_Q(\operatorname{Hom}_Q(M,L),N) \arrow[d, "\simeq"] \\
\operatorname{Hom}_Q(L',Q)\otimes_Q M'\otimes_Q N \arrow[r, "\cong", "\theta"'] & \operatorname{Hom}_Q(\operatorname{Hom}_Q(M',L'),N)
\end{tikzcd}$$

It is clear that θ in the lower row is bijective when L' and M' are shifts of Q. The case when L' and M' are modules follows, as the functors involved commute with finite direct sums. Finally, when L' and M' are bounded complexes the assertion is obtained by induction on the number of the degrees in which they are not zero.

(2) There is a quasi-isomorphism $N\simeq N'$ with N' a bounded complex of flat Q-modules. It is preserved by tensoring with arbitrary complexes, and this explains the vertical arrows in the commutative diagram of morphisms of complexes

$$\begin{tikzcd}
\operatorname{Hom}_Q(L,Q)\otimes_Q N \arrow[r, "\vartheta"] \arrow[d, dash, "\simeq"'] & \operatorname{Hom}_Q(L,N) \arrow[d, dash, "\simeq"] \\
\operatorname{Hom}_Q(L,Q)\otimes_Q N' \arrow[r, "\cong", "\vartheta"'] & \operatorname{Hom}_Q(L,N')
\end{tikzcd}$$

The map ϑ in the lower row is bijective because the complex N' is bounded and each Q-module L_i is finite projective. □

8.9. Let Q be a ring and M,N complexes of Q-modules.

Choosing a semiprojective resolution $P\to M$, one sets

$$M\otimes^{\mathbf{L}}_Q N = P\otimes_Q N \qquad\text{and}\qquad \mathbf{R}\operatorname{Hom}_Q(M,N)=\operatorname{Hom}_Q(P,N)\,.$$

In $\mathsf{D}(Q)$, the derived category of Q-modules, these complexes are well-defined up to isomorphism. If $N\to I$ is a semiinjective resolution, the natural morphisms

$$\operatorname{Hom}_Q(P,N)\to\operatorname{Hom}_Q(P,I)\leftarrow\operatorname{Hom}_Q(M,I)\,.$$

are quasi-isomorphisms. For each integer i, one sets

$$\operatorname{Tor}_i^Q(M,N) = \mathrm{H}_i(M \otimes^{\mathbf{L}}_Q N) \quad \text{and} \quad \operatorname{Ext}^i_Q(M,N) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_Q(M,N))\,.$$

When M and N are Q-modules these are the usual (co)homology modules.

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