

## RESOLUTIONS OF MONOMIAL IDEALS AND COHOMOLOGY OVER EXTERIOR ALGEBRAS

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**ABSTRACT.** The paper studies the homology of finite modules over the exterior algebra  $E$  of a vector space  $V$ . To such a module  $M$  we associate an algebraic set  $V_E(M) \subseteq V$ , consisting of those  $v \in V$  that have a non-minimal annihilator in  $M$ . A cohomological description of its defining ideal leads, among other things, to complementary expressions for its dimension, linked by a ‘depth formula’. Explicit results are obtained for  $M = E/J$ , when  $J$  is generated by products of elements of a basis  $e_1, \dots, e_n$  of  $V$ . A (infinite) minimal free resolution of  $E/J$  is constructed from a (finite) minimal resolution of  $S/I$ , where  $I$  is the squarefree monomial ideal generated by ‘the same’ products of the variables in the polynomial ring  $S = K[x_1, \dots, x_n]$ . It is proved that  $V_E(E/J)$  is the union of the coordinate subspaces of  $V$ , spanned by subsets of  $\{e_1, \dots, e_n\}$  determined by the Betti numbers of  $S/I$  over  $S$ .

### INTRODUCTION

Let  $V$  be a vector space with basis  $e_1, \dots, e_n$  over a field  $K$ , and let  $E = \bigwedge(V)$  be the exterior algebra over  $V$ . The standard basis elements  $e_{k_1} \wedge \dots \wedge e_{k_s}$  of  $E$ ,  $k_1 < \dots < k_s$ , are called monomials in  $E$ . An ideal  $J \subseteq E$  generated by monomials is called a monomial ideal. We study the (co)homological algebra of such ideals.

Along with  $J$ , we consider the corresponding squarefree monomial ideal  $I$  in the polynomial ring  $S = K[x_1, \dots, x_n]$ . Each  $S$ -module  $F_i$  in a minimal multigraded free resolution  $F$  of  $S/I$  can be written in the form

$$F_i = \bigoplus_{j=1}^{\beta_i} S(-a_{ij}) \quad \text{with uniquely determined } a_{ij} \in \mathbb{N}^n.$$

A well known formula of Hochster [12] on the multigraded Betti numbers of square-free monomial ideals shows that  $F$  is itself squarefree, in the sense that the coordinates of all shifts  $a_{ij}$  are equal to 0 or 1. Furthermore, there exist interesting non-minimal squarefree resolutions, for example the Taylor resolution [15].

Given any squarefree resolution  $F$  of the monomial ideal  $I \subseteq S$ , we choose a homogeneous basis  $B$  of  $F$  and construct a multigraded free resolution  $G$  of the monomial ideal  $J$  in the exterior algebra  $E$ . The resolution depends on  $B$ , but different choices of multihomogeneous bases lead to isomorphic complexes; if  $F$  is minimal, then so is  $G$ . The construction is given in Section 1.

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Section 2 contains applications. An explicit formula gives the multigraded Betti numbers of the monomial ideal  $J \subseteq E$  in terms of those of  $I$ . As a consequence, some interesting properties of  $J$ , like the linearity of its minimal resolution or the independence of its Betti numbers from the characteristic of the base field  $K$ , are seen to be equivalent to the corresponding properties of  $I$ . We also show that if  $I$  is a Gotzmann ideal in  $S$ , then  $J$  is a Gotzmann ideal in  $E$ . Our method yields exterior algebra analogues of the Taylor [15] and Eliahou-Kervaire [10] resolutions.

In Section 3 we associate with each finite  $E$ -module  $M$  an algebraic set  $V_E(M) \subseteq V$ . As for modular representations of finite groups, which provide the model, there are two constructions: in terms of the action of the graded ring  $\text{Ext}_E(K, K)$  on  $\text{Ext}_E(M, K)$ , following Quillen [14], or in terms of the action of  $V$  on  $M$ , mimicking Carlson [7]. We prove that they yield the same result. Along with other properties of  $V_E(M)$ , this parallels results over group algebras; techniques developed for that case have been successfully extended to other Hopf algebras, but they do not always apply here, because  $E$  is *not* a Hopf algebra (in the category of rings). Our approach is similar to that used in [4] to study modules over complete intersections, and takes advantage of the simple structure of  $\text{Ext}_E(K, K)$ : by Cartan [8] it is the symmetric algebra of  $\text{Hom}_K(V, K)$ . In particular, we prove that the dimension of  $V_E(M)$  is complementary to the (appropriately defined) depth of  $M$  over  $E$ .

When  $\Delta$  is a simplicial complex and  $J = J_\Delta$  is the ideal in  $E$  generated by all monomials  $e_{k_1} \wedge \cdots \wedge e_{k_s}$  such that  $\{k_1, \dots, k_s\} \not\subseteq \Delta$ , the  $K$ -algebra  $K\langle\Delta\rangle = E/J_\Delta$  is called the indicator algebra of  $\Delta$ . It has proved to be important in the study of the  $f$ -vector of  $\Delta$ ; see for example [3]. The corresponding squarefree ideal  $I = I_\Delta$  in  $S$  defines the more familiar Stanley-Reisner ring  $K[\Delta] = S/I_\Delta$ . In Section 4 we prove that  $V_E(K\langle\Delta\rangle)$  is a union of coordinate subspaces of  $V$ , determined by the supports of the shifts of a minimal free resolution of the Stanley-Reisner ring  $K[\Delta]$  over  $S$ . This has consequences for the simplicial cohomology of  $\Delta$ .

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## 1. THE MAIN CONSTRUCTION

For the rest of the paper we fix some—mostly standard—notation.

A  $n$ -tuple  $(a_1, \dots, a_n) \in \mathbb{Z}^n$  is *squarefree* if  $0 \leq a_j \leq 1$  for  $j = 1, \dots, n$ . For  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$  we set  $|a| = a_1 + \cdots + a_n$ , and  $\text{supp}(a) = \{j \mid a_j \neq 0\}$ ; by convention,  $\text{supp}(0) = \emptyset$ , where  $[n] = \{1, \dots, n\}$ . For an element  $u$  of an  $n$ -graded vector space  $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$ , the notation  $\text{deg}(u) = a$  is equivalent to  $u \in M_a$ ; we set  $\text{supp}(\text{deg}(u)) = \text{supp}(u)$  and  $|\text{deg}(u)| = |u|$ . The decomposition  $M = \bigoplus_{j \in \mathbb{Z}} M_j$ , where  $M_j = \bigoplus_{a \in \mathbb{Z}^n, |a|=j} M_a$ , turns  $M$  into a graded vector space.

Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring on  $n$  commuting variables, and let  $E = K\langle e_1, \dots, e_n \rangle$  be the exterior algebra on  $n$  alternating variables. They are  $n$ -graded by  $\text{deg}(x_j) = \text{deg}(e_j) = \varepsilon_j = (0, \dots, 0, 1, 0, \dots, 0)$ , with 1 in  $j$ th position. For  $\sigma \subseteq [n]$  we set  $x^\sigma = x_{k_1} \cdots x_{k_s}$  and  $e_\sigma = e_{k_1} \wedge \cdots \wedge e_{k_s}$ , where  $\sigma = \{k_1, \dots, k_s\}$  with  $k_1 < \cdots < k_s$ ; we say that  $e_\sigma$  is a *monomial* in  $E$ . For  $a \in \mathbb{N}^n$  we set  $x^a = x_1^{a_1} \cdots x_n^{a_n}$  and  $e_a = e_{\text{supp}(a)}$ .

The following simple observation is used in many computations.

**Observation 1.0.** For monomials  $u, v \in E$  with  $\text{supp}(v) \subseteq \text{supp}(u)$  there exists a unique monomial  $u' \in E$  such that  $vu' = u$ ; we then set  $v^{-1}u = u'$ . For monomials  $u, v, w, z \in E$  the equalities below hold whenever the left hand side is defined:

$$(v^{-1}u)w = v^{-1}(uw) \quad \text{and} \quad (z^{-1}v)(v^{-1}u) = z^{-1}u.$$

**Construction 1.1.** Let  $(F, \theta)$  be a *squarefree complex* of  $n$ -graded  $S$ -modules, meaning that each  $F_i$  has a basis  $B_i$  with  $\deg(f)$  squarefree for all  $f \in B_i$ .

Let  $P_i$  be a  $n$ -graded  $K$ -vector space with basis  $B_i$ , and set  $B = \bigsqcup_i B_i$ . Let  $C_j$  be the  $n$ -graded right  $E$ -module with basis  $\{y^{(a)} \mid a \in \mathbb{N}^n, \deg(y^{(a)}) = a, |a| = j\}$ . The tensor product  $C_j \otimes_K P_i$  becomes a right  $n$ -graded  $E$ -module, by

$$\begin{aligned} \deg(y^{(a)} \otimes f) &= a + b; \\ (y^{(a)} \otimes f)e &= (-1)^{|b|} y^{(a)} e \otimes f, \end{aligned} \quad \text{where } b = \deg(f).$$

Let  $G_\ell$  be the residue module of  $\bigoplus_{\ell=j+i} C_j \otimes_K P_i$  by the submodule generated by  $\{y^{(a)} \otimes f \mid \text{supp}(a) \not\subseteq \text{supp}(f)\}$ , and write  $y^{(a)}f$  for the image of  $y^{(a)} \otimes f$  in  $G_\ell$ . Thus,  $G_\ell$  is the  $n$ -graded right  $E$ -module with basis

$$Y_\ell = \left\{ y^{(a)}f \mid \begin{array}{l} a \in \mathbb{N}^n, f \in B_i, \text{supp}(a) \subseteq \text{supp}(f) \\ \ell = |a| + i, \deg(y^{(a)}f) = a + \deg(f) \end{array} \right\}$$

If in the complex  $(F, \theta)$  the differential of  $f \in B_i$  has the form

$$\theta(f) = \sum_{j: f_j \in B_{i-1}} \lambda_j x^{b-b_j} f_j \quad \text{with } \lambda_j \in K, b = \deg(f), b_j = \deg(f_j),$$

then define homomorphisms  $G_\ell \rightarrow G_{\ell-1}$  of  $n$ -graded  $E$ -modules by

$$\begin{aligned} \delta(y^{(a)}f) &= (-1)^{|b|} \sum_{k \in \text{supp}(a)} y^{(a-\varepsilon_k)} f e_k \\ \vartheta(y^{(a)}f) &= (-1)^{|a|} \sum_{j: f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b. \end{aligned}$$

and set  $\partial = \delta + \vartheta: G_\ell \rightarrow G_{\ell-1}$ .

**Proposition 1.2.** *The preceding construction yields a complex  $(G, \partial)$  of right  $n$ -graded  $E$ -modules. If  $(G', \partial')$  is the complex obtained from homogeneous bases  $B'_i$  of  $F_i$ , then  $G' \cong G$  as complexes of  $n$ -graded  $E$ -modules.*

Hochster's formula [12] for the Betti numbers of a squarefree monomial ideal  $I \subseteq S$  shows that its minimal free resolution  $(F, \theta)$  is squarefree. In that case, we can say more about the complex  $(G, \partial)$  described above.

**Theorem 1.3.** *Let  $\Sigma$  be a set of subsets of  $[n]$ , let  $I \subseteq S = K[x_1, \dots, x_n]$  be the ideal generated by the squarefree monomials  $\{x^\sigma \mid \sigma \in \Sigma\}$ , and let  $J \subseteq E = K\langle e_1, \dots, e_n \rangle$  be the ideal generated by the monomials  $\{e_\sigma \mid \sigma \in \Sigma\}$ .*

*If  $(F, \theta)$  is a (minimal) free resolution of  $S/I$  over  $S$ , then the complex  $(G, \partial)$  of Construction 1.1 is a (minimal) free resolution of  $E/J$  over  $E$ .*

*Proof of the proposition.* To show that  $\partial^2 = 0$  we establish equalities

$$\delta^2 = 0; \quad \vartheta^2 = 0; \quad \delta\vartheta = -\vartheta\delta.$$

The first one comes from an easy direct computation.

Writing  $\theta(f_j) = \sum_{k: g_k \in B_{i-2}} \mu_{kj} x^{b_j - c_k} g_k \in F_{i-2}$ , we have

$$\begin{aligned} \theta^2(f) &= \sum_j \lambda_j x^{b-b_j} \theta(f_j) = \sum_j \lambda_j x^{b-b_j} \sum_k \mu_{kj} x^{b_j - c_k} g_k \\ &= \sum_k \left( \sum_j \mu_{kj} \lambda_j \right) x^{b-c_k} g_k = 0. \end{aligned}$$

Thus,  $\sum_j \mu_{kj} \lambda_j = 0$ , so we get the second equality from:

$$\begin{aligned} \vartheta^2(y^{(a)} f) &= (-1)^{|a|} \sum_j \vartheta(y^{(a)} f_j) (\lambda_j e_{b_j}^{-1} e_b) \\ &= \sum_j \left( \sum_k y^{(a)} g_k (\mu_{kj} e_{c_k}^{-1} e_{b_j}) \right) (\lambda_j e_{b_j}^{-1} e_b) \\ &= \sum_k y^{(a)} g_k \left( \sum_j \mu_{kj} \lambda_j (e_{c_k}^{-1} e_{b_j}) (e_{b_j}^{-1} e_b) \right) \\ &= \sum_k y^{(a)} g_k \left( \sum_j \mu_{kj} \lambda_j \right) e_{c_k}^{-1} e_b = 0. \end{aligned}$$

Note that if  $f \in B$  with  $\deg(f) = b$  and  $e \in E$  with  $\deg(e) = c$ , then

$$\begin{aligned} \delta(y^{(a)} f e) &= \delta(y^{(a)} f) e \\ \vartheta(y^{(a)} f e) &= \vartheta(y^{(a)} f) e \end{aligned} \quad \text{provided } \text{supp}(a) \subseteq \text{supp}(b) + \text{supp}(c).$$

When  $\text{supp}(a) \subseteq \text{supp}(b)$  these formulas hold by definition. If  $\text{supp}(a) \not\subseteq \text{supp}(b)$ , then  $y^{(a)} f = 0$ , so we check that the right hand sides vanish. On the one hand,  $\delta(y^{(a)} f e) = \pm \sum_{k \in \text{supp}(a)} y^{(a-\varepsilon_k)} f e_k e$ ; if  $\text{supp}(a-\varepsilon_k) \not\subseteq \text{supp}(b)$ , then  $y^{(a-\varepsilon_k)} f = 0$ ; otherwise,  $k \in \text{supp}(a) \setminus \text{supp}(f)$ , hence  $k \in \text{supp}(c)$ , so  $e_k e = 0$ . On the other hand,  $\vartheta(y^{(a)} f) = \pm \sum_j y^{(a)} g_j (\lambda_j e_{b_j}^{-1} e_b)$  with  $g_j \in B$ . Since  $\text{supp}(g_j) \subseteq \text{supp}(f)$ , for all  $j$  we have  $\text{supp}(a) \not\subseteq \text{supp}(g_j)$ , and hence  $y^{(a)} g_j = 0$ .

The third equality now results from the computation:

$$\begin{aligned} \vartheta(\delta(y^{(a)} f)) &= (-1)^{|b|} \vartheta \left( \sum_{k \in \text{supp}(a)} y^{(a-\varepsilon_k)} f e_k \right) = (-1)^{|b|} \sum_{k \in \text{supp}(a)} \vartheta(y^{(a-\varepsilon_k)} f) e_k \\ &= (-1)^{|b|+|a|-1} \sum_{k \in \text{supp}(a)} \left( \sum_{j: f_j \in B_{i-1}} y^{(a-\varepsilon_k)} f_j \lambda_j e_{b_j}^{-1} e_b \right) e_k \\ &= (-1)^{|a|-1} \sum_{j: f_j \in B_{i-1}} \left( \sum_{k \in \text{supp}(a)} (-1)^{|b_j|} y^{(a-\varepsilon_k)} f_j e_k \right) \lambda_j e_{b_j}^{-1} e_b \\ &= (-1)^{|a|-1} \sum_{j: f_j \in B_{i-1}} \delta(y^{(a)} f_j) \lambda_j e_{b_j}^{-1} e_b \\ &= (-1)^{|a|-1} \delta \left( \sum_{j: f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b \right) = -\delta(\vartheta(y^{(a)} f)). \end{aligned}$$

When  $(G', \vartheta')$  is a complex obtained from a homogeneous basis  $B'$  of  $F$ , write each  $f' \in B'_i$  in the form  $f' = \sum_{j: f_j \in B_i} \lambda_j x^{b'-b_j} f_j$  with  $b' = \deg(f')$  and  $b_j = \deg(f_j)$ , and define homomorphisms of  $E$ -modules  $\gamma_i: G'_i \rightarrow G_i$  by

$$\gamma_i(y^{(a)} f') = \sum_{j: f_j \in B_i} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_{b'}.$$

Computations similar to (and more straightforward than) those above show that  $\gamma(\vartheta'(y^{(a)} f')) = \vartheta(\gamma(y^{(a)} f'))$  and  $\gamma(\delta'(y^{(a)} f')) = \delta(\gamma(y^{(a)} f'))$ , so  $\gamma$  is a chain map. It is clearly bijective, so we have the desired isomorphism.  $\square$

*Proof of the theorem.* Let  $(F, \theta)$  be a  $n$ -graded free resolution of  $S/I$  over  $S$ , and let  $(G, \partial)$  be the complex obtained from it by Construction 1.1. To show that it is a resolution of  $E/J$ , we construct a  $K$ -linear chain homotopy  $\chi$  such that

$$\chi\partial + \partial\chi = \text{id}_{\tilde{G}} \quad (*)$$

where  $\tilde{G}$  is the complex obtained from  $G$  by replacing  $G_0$  with  $J$ .

Since  $F$  is exact, there is a homogeneous  $K$ -linear chain homotopy  $\tau$  such that

$$\tau\theta + \theta\tau = \text{id}_{\tilde{F}}$$

where  $\tilde{F}$  is the complex obtained from  $F$  by replacing  $F_0$  with  $I$ .

Thus, for  $f \in B$  with  $\deg(f) = b$  and  $\sigma \subseteq [n]$  such that  $\text{supp}(b) \cap \sigma = \emptyset$ , we have

$$\tau(fx^\sigma) = \sum_k \mu_k x^\sigma x^{b-a_k} h_k \quad \text{where } \mu_k \in K, h_k \in B, a_k = \deg(h_k).$$

We define a  $K$ -linear map  $\chi$  on the  $K$ -basis of  $\tilde{G}$  described in Construction 1.1 by

$$\chi(y^{(a)} f e_\sigma) = \begin{cases} \sum_k h_k \mu_k e_{a_k}^{-1}(e_b e_\sigma) & \text{if } a = 0 \text{ and } \text{supp}(b) \cap \sigma = \emptyset & (1) \\ (-1)^{r+|b|} y^{\varepsilon_s} f e_{\sigma \setminus \{s\}} & \text{if } a = 0 < \min(\text{supp}(b) \cap \sigma) = s & (2) \\ 0 & \text{if } a \neq 0 \text{ and } \text{supp}(b) \cap \sigma = \emptyset & (3) \\ 0 & \text{if } 0 < \min(a) < \min(\text{supp}(b) \cap \sigma) & (4) \\ (-1)^{r+|b|} y^{(a+\varepsilon_s)} f e_{\sigma \setminus \{s\}} & \text{if } \min(a) \geq \min(\text{supp}(b) \cap \sigma) = s & (5) \end{cases}$$

where  $b = \text{supp}(f)$  and  $r = |\{k \in \sigma \mid k < \min(\text{supp}(b) \cap \sigma)\}|$ .

We establish  $(*)$ , by four separate computations. To simplify notation, we set

$$s(c) = \text{supp}(c) \quad \text{for } c \in \mathbb{N}^n \quad \text{and} \quad u_j = \lambda_j e_{b_j}^{-1} e_b \quad \text{for } j \in [n].$$

(1) One has  $\partial(fe_\sigma) = \sum_j f_j(u_j)e_\sigma$ . Since  $s(u_j) = s(b) \setminus s(b_j)$  for every  $j$ , we get

$$s(u_j) \cap \sigma = \emptyset \quad \text{and} \quad s(b_j) \cap s(u_j e_\sigma) = s(b_j) \cap \sigma = \emptyset.$$

Write  $\tau(f_j x^\sigma x^{b-b_j}) = \sum_\ell g_\ell \nu_{\ell j} x^\sigma x^{b-c_\ell}$  with  $g_\ell \in B$ ,  $\nu_{\ell j} \in K$  and  $c_\ell = \deg(g_\ell)$ . As  $e_{b_j} u_j e_\sigma = \lambda_j e_b e_\sigma$ , one has  $\chi(f_j u_j e_\sigma) = \lambda_j \sum_\ell g_\ell \nu_{\ell j} e_{c_\ell}^{-1}(e_b e_\sigma)$ , therefore

$$\chi(\partial(fe_\sigma)) = \sum_\ell g_\ell \left( \sum_j \lambda_j \nu_{\ell j} \right) e_{c_\ell}^{-1}(e_b e_\sigma).$$

On the other hand, if  $\theta(h_k) = \sum_\ell g_\ell \lambda_{\ell k} x^{a_k - c_\ell}$  with  $\lambda_{\ell k} \in K$ , then

$$\partial(\chi(fe_\sigma)) = \sum_k \sum_\ell g_\ell \mu_k \lambda_{\ell k} (e_{c_\ell}^{-1} e_{a_k})(e_{a_k}^{-1}(e_b e_\sigma)) = \sum_\ell g_\ell \left( \sum_k \mu_k \lambda_{\ell k} \right) e_{c_\ell}^{-1}(e_b e_\sigma).$$

Since  $\theta\tau + \tau\theta = \text{id}_F$ , we see that there exists a  $\ell_0$  such that  $g_{\ell_0} = f$ , and

$$\sum_k \mu_k \lambda_{\ell_0 k} + \sum_j \lambda_j \nu_{\ell_0 j} = \begin{cases} 1 & \text{if } \ell = \ell_0; \\ 0 & \text{if } \ell \neq \ell_0. \end{cases}$$

This shows that  $\partial\chi(fe_\sigma) + \chi\partial(fe_\sigma) = fe_\sigma$ , as desired.

(2) and (5). In either case,  $\partial\chi(y^{(a)}fe_\sigma)$  is equal to

$$(-1)^r \sum_{k \in s(a+\varepsilon_s)} y^{(a+\varepsilon_s-\varepsilon_k)} fe_k e_{\sigma \setminus \{s\}} + (-1)^{r+|b|+|a|+1} \sum_{j: s(b_j) \supseteq s(a+\varepsilon_s)} y^{(a+\varepsilon_s)} f_j u_j e_{\sigma \setminus \{s\}}.$$

Note that  $y^{(a)}fe_\sigma$  appears above as a summand in the first sum for  $k = s$ . Now we compute  $\chi(\partial(y^{(a)}fe_\sigma))$ . If  $s \notin s(b_j)$  for some  $j$ , then  $s \in s(b) \setminus s(b_j) = s(u_j)$ , therefore  $u_j e_s = 0$ , so that in  $\partial(y^{(a)}fe_\sigma)$  only the summands  $y^{(a)}f_j u_j e_\sigma$  with  $s \in s(b_j)$  remain. In this case  $\min(s(b_j) \cap s(u_j e_\sigma)) = s$ , hence

$$\chi(y^{(a)}f_j u_j e_\sigma) = (-1)^{r+|b_j|+|u_j|} y^{(a+\varepsilon_s)} f_j u_j e_{\sigma \setminus \{s\}}.$$

Since  $|u_j| + |b_j| = |b|$ , we see that the second sum in  $\partial\chi(y^{(a)}fe_\sigma)$  appears in  $\chi(\partial(y^{(a)}fe_\sigma))$  with the opposite sign. If  $k \in s(a)$  and  $k \notin \sigma$ , then  $k \geq \min(a) \geq s$ , so  $\min(s(b) \cap (\sigma \cup k)) = s$ . As  $\min(a - \varepsilon_k) \geq \min(a) \geq s$ , we get

$$(-1)^{|b|} \chi(y^{(a-\varepsilon_k)} fe_k e_\sigma) = (-1)^{r+1} y^{(a+\varepsilon_s-\varepsilon_k)} fe_k e_{\sigma \setminus \{s\}}.$$

The desired equality follows.

(3) For each  $j$  with  $s(a) \subseteq s(b_j)$ , one has  $s(b_j) \cap \sigma = \emptyset$ , hence  $\chi(y^{(a)}f_j u_j e_\sigma) = 0$ . Let  $k \in s(a)$ ,  $k \notin \sigma$  and consider  $\chi(y^{(a-\varepsilon_k)} fe_k e_\sigma)$ . We have now  $s(b) \cap (\sigma \cup k) = k$ . If  $k > \min(a)$ , then  $\min(a - \varepsilon_k) = \min(a)$ , therefore  $\chi(y^{(a-\varepsilon_k)} fe_k e_\sigma) = 0$ . Let  $k = \min(a)$ . Then  $\min(a - \varepsilon_k) \geq k$ , hence  $(-1)^{|b|} \chi(y^{(a-\varepsilon_k)} fe_k e_\sigma) = y^{(a)}fe_\sigma$ . This proves the desired equality.

(4) For each  $j$  with  $s(a) \subseteq s(b_j)$ , one has  $u_j e_m = 0$  or  $\min(s(b_j) \cap \sigma) = m$ , so that in both cases  $\chi(y^{(a)}f_j u_j e_\sigma) = 0$ . Let  $k \in s(a)$ ,  $k \notin \sigma$  and consider  $\chi(y^{(a-\varepsilon_k)} fe_k e_\sigma)$ . If  $k > \min(a)$ , then  $\min(a - \varepsilon_k) = \min(a) < m$ , therefore  $\min(a) < \min(s(b) \cap (\sigma \cup k))$  and by definition  $\chi(y^{(a-\varepsilon_k)} fe_k e_\sigma) = 0$ . Let  $k = \min(a)$ . Then  $\min(s(b) \cap (\sigma \cup k)) = k \leq \min(a - \varepsilon_k)$ , therefore  $(-1)^{|b|} \chi(y^{(a-\varepsilon_k)} fe_k e_\sigma) = y^{(a)}fe_\sigma$ . This proves (\*).  $\square$

## 2. APPLICATIONS

Recall that each finite  $n$ -graded module  $M$  over  $A = E$  or  $A = S$  has a unique up to isomorphism minimal resolution by free  $n$ -graded  $A$ -modules, and homogeneous  $A$ -linear homomorphisms. The *multigraded Betti number*  $\beta_{ia}^A(M)$  is the number of basis elements of the  $i$ 'th free module in such a resolution, that are homogeneous of degree  $a$ . The *multigraded Poincaré series* of  $M$  over  $A$  is defined by

$$P_M^A(t, u) = \sum_{i \geq 0} \sum_{a \in \mathbb{N}^n} \beta_{ia}^A(M) t^i u^a.$$

For the rest of this section,  $I$  is an ideal generated by squarefree monomials in  $S$ , and  $J$  denotes the corresponding monomial ideal in  $E$ .

Counting ranks in the resolution of Theorem 1.3 we get a new proof of [3, (6.4)].

**Proposition 2.1.** *There is an equality of formal power series*

$$P_{E/J}^E(t, u) = \sum_{i \geq 0} \sum_{a \in \mathbb{N}^n} \beta_{ia}^S(S/I) \frac{t^i u^a}{\prod_{j \in \text{supp}(a)} (1 - tu_j)}. \quad \square$$

We record a couple of immediate consequences of this formula.

**Corollary 2.2.** (1) *The multigraded Betti numbers of  $I$  are independent of the characteristic of the field  $K$  if and only if this is true for  $J$ .*

(2) *The ideal  $I$  has a linear free resolution over  $S$  if and only if the ideal  $J$  has a linear free resolution over  $E$ .*  $\square$

An important class of ideals in  $S$  with linear resolution are the Gotzmann ideals.

Recall that an ideal  $L \subseteq A$ , where  $A = S$  or  $A = E$ , is called *Gotzmann* if it is generated by elements of the same degree, say  $d$ , and its span in degree  $d+1$  is the smallest possible:  $\text{rank}_K L_{d+1} \leq \text{rank}_K L'_{d+1}$  holds for all graded ideals  $L' \subseteq A$  with  $\text{rank}_K L'_d = \text{rank}_K L_d$ . It is a widely open question which monomial ideals are Gotzmann. From a combinatorial point of view, it is particularly interesting for ideals generated by squarefree monomials.

**Proposition 2.3.** *If the ideal  $I \subseteq S$  is Gotzmann, then so is the ideal  $J \subseteq E$ .*

Note that the converse may fail:  $J = (e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4) \subseteq E$  is a Gotzmann ideal, but  $I = (x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4) \subseteq S$  is not.

*Proof.* Let  $J' \subseteq E$  be an ideal generated in degree  $d$ , with  $\text{rank}_K J'_d = \text{rank}_K J_d$ .

The algebraic Kruskal-Katona Theorem [3, (4.4)] yields a monomial ideal  $J^{\text{lex}}$  generated in degree  $d$ , with  $\text{rank}_K J_d^{\text{lex}} = \text{rank}_K J'_d$  and  $\text{rank}_K J_{d+1}^{\text{lex}} \leq \text{rank}_K J'_{d+1}$ . For the squarefree monomial ideal  $I' \subseteq S$  corresponding to  $J^{\text{lex}}$ , we have

$$\begin{aligned} \text{rank}_K J_{d+1} &= n\beta_{0d}(J) - \beta_{1d+1}(J) \\ &= n\beta_{0d}(I) - (\beta_{1d+1}(I) + d\beta_{0d}(I)) \\ &= \text{rank}_K I_{d+1} - d\text{rank}_K I_d \\ &\leq \text{rank}_K I'_{d+1} - d\text{rank}_K I'_d \\ &= n\beta_{0d}(I') - (\beta_{1d+1}(I') + d\beta_{0d}(I')) \\ &= n\beta_{0d}(J^{\text{lex}}) - \beta_{1d+1}(J^{\text{lex}}) \\ &= \text{rank}_K J_{d+1}^{\text{lex}} \end{aligned}$$

where the inequality is the Gotzmann hypothesis on  $I$ , the second and penultimate equalities come from Proposition 2.1, the rest are read off from the corresponding minimal resolutions. Altogether, we get  $\text{rank}_K J_{d+1} \leq \text{rank}_K J'_{d+1}$ , as desired.  $\square$

Applying Theorem 1.3 to the Taylor resolution of monomial ideals in polynomial rings, cf. [15] or [9, p. 439], we obtain an analogue over exterior algebras.

For a set of monomials  $\{u_1, \dots, u_m\}$  and a subset  $\tau \subseteq [m] = \{1, \dots, m\}$ , we denote  $u_\tau$  the least common multiple of the monomials  $\{u_j \mid j \in \tau\}$ .

**Proposition 2.4.** *Let  $J \subseteq E$  be an ideal generated by a set  $\{u_1, \dots, u_m\}$  of monomials. The right  $E$ -modules  $T_i$  with basis*

$$\{y^{(a)} f_\tau \mid a \in \mathbb{N}^n, |a| + |\tau| = i, \tau \subseteq [m], \text{supp}(a) \subseteq \text{supp}(u_\tau)\}$$

where  $\deg(y^{(a)} f_\tau) = a + \deg u_\tau$ , and the  $E$ -linear maps defined by

$$\begin{aligned} \partial(y^{(a)} f_\tau) &= (-1)^{|u_\tau|} \sum_{k \in \text{supp}(a)} y^{(a - \varepsilon_k)} f_\tau e_k + \\ &\quad \sum_{j: \text{supp}(u_{\tau \setminus \{j\}}) \supseteq \text{supp}(a)} (-1)^{r_j + |a|} y^{(a)} f_{\tau \setminus \{j\}} u_{\tau \setminus \{j\}}^{-1} u_\tau \end{aligned}$$

where  $r_j = |\{t \in \tau \mid t < j\}|$ , form a  $n$ -graded resolution of  $E/J$ .  $\square$

**Example 2.5.** When  $J = (e_1, \dots, e_n)$  the proposition provides a minimal  $n$ -graded resolution of  $K = E/(e_1, \dots, e_n)$  over  $E$ . Another one is the *Cartan resolution*  $(C, \partial)$ , where  $C_i$  has a basis  $\{w^{(c)} \mid c \in \mathbb{N}^n, |c| = i\}$ , and

$$d(w^{(c)}) = \sum_{k \in \text{supp}(c)} w^{(c - \varepsilon_k)} e_k.$$

To get an isomorphism of complexes  $\gamma: C \rightarrow T$ , note that each  $c \in \mathbb{N}^n$  can be written uniquely as  $c = a + b$  with  $\text{supp}(c) = \text{supp}(b)$  and  $b$  squarefree, and set

$$\gamma(w^{(c)}) = (-1)^{|b|(|a| + (|b|-1)/2)} y^{(a)} f_{\text{supp}(b)}.$$

Our last application is to stable ideals, a notion extended in [3] from polynomial rings to exterior algebras: setting  $\max(e_\sigma) = \max\{i \mid i \in \sigma\}$ , call a monomial ideal  $J \subseteq E$  *stable* if  $e_j e_{\sigma \setminus \{m\}} \in J$  for each  $e_\sigma \in J$  and each  $j < m = \max(e_\sigma)$ .

For a monomial ideal  $J \subseteq E$ , we denote  $G(J)$  the uniquely defined minimal generating set of  $J$  consisting of monomials. As in [10], it is easily seen that each monomial  $u' \in J$  has a unique decomposition  $u' = uw$  with  $u \in G(J)$  and  $\max(u) < \min(w)$ . Applying Theorem 1.3 to the resolution of squarefree stable ideals in  $S$  given in [2], we get a resolution for stable monomial ideals in  $E$ .

**Proposition 2.6.** *If  $J \subseteq E$  is a stable ideal, then  $E/J$  has a minimal resolution  $(G, \partial)$  by  $n$ -graded free  $E$ -modules  $G_\ell$  with basis*

$$\left\{ y^{(a)} f_{\sigma, u} \mid \begin{array}{l} a \in \mathbb{N}^n, \sigma \subseteq [n], u \in G(J) \\ \text{supp}(a) \subseteq \sigma \cup \text{supp}(u), \sigma \cap \text{supp}(u) = \emptyset, \max(\sigma) < \max(u) \\ i = |a| + |\sigma| + 1, \deg(y^{(a)} f_{\sigma, u}) = a + \deg(e_\sigma) + \deg(u) \end{array} \right\}$$

and differentials  $\partial_\ell: G_\ell \rightarrow G_{\ell-1}$  given by

$$\begin{aligned} \partial(y^{(a)} f_{\sigma, u}) &= (-1)^{|u| + |\sigma|} \sum_{\ell \in \text{supp}(a)} y^{(a - \varepsilon_\ell)} f_{\sigma, u} e_\ell + \\ &\quad (-1)^{|a|} \sum_{j \in \sigma} \left( (-1)^{|\sigma|} y^{(a)} f_{\sigma \setminus \{j\}, u} e_j + (-1)^{(|\sigma|-1)|w_j|} f_{\sigma \setminus \{j\}, u_j} w_j \right) \end{aligned}$$

where  $u_j \in G(J)$  is determined from the unique decomposition  $u e_j = u_j w_j$  described above, and  $y^{(b)} f_{\rho, v} = 0$  if  $\max(\rho) > \max(v)$  or  $\text{supp}(b) \not\subseteq \rho \cup \text{supp}(v)$ .  $\square$

The preceding result was originally proved by different means in [3, (2.1)].

### 3. COHOMOLOGY

We study right modules over the exterior algebra  $E$ . Since the ideal  $(V) \subset E$  is nilpotent, each (finite)  $E$ -module  $M$  has a unique up to isomorphism minimal free resolution  $F$  by (finite) free  $E$ -modules. The rank  $\beta_i^E(M)$  of the free  $E$ -module  $F_i$  is known as the  $i$ 'th *Betti number* of  $M$  over  $E$ . The size of  $F$  is measured by the *complexity* of  $M$  over  $E$ , that is introduced as follows:

$$\text{cx}_E M = \inf\{c \in \mathbb{Z} \mid \beta_i^E(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and all } i \geq 1\}.$$

For each  $v \in V = E_1$ , the equality  $v^2 = 0$  implies  $Mv \subseteq \text{Ann}_M(v)$ . We say that  $v$  is  $M$ -*regular* if equality holds, or, equivalently, if the infinite complex of  $K$ -spaces

$$(M, \rho^v): \quad \dots \rightarrow M \xrightarrow{\rho^v} M \xrightarrow{\rho^v} M \rightarrow \dots \quad \text{where} \quad \rho^v(y) = yv$$

has trivial homology  $H_*(M, \rho^v)$ . Otherwise, we say that  $v$  is  $M$ -*singular*.

The set  $V_E(M) \subseteq V$  of  $M$ -singular elements is called the *rank variety* of  $M$ .

If  $M = \bigoplus_{a \in \mathbb{Z}} M_a$  is *graded*, regularity can also be introduced by the vanishing of the cohomology  $H^*(M, v)$  of the finite complex of  $K$ -vector spaces

$$(M, v): \quad \dots \rightarrow M_{a-1} \xrightarrow{\rho_{a-1}^v} M_a \xrightarrow{\rho_a^v} M_{a+1} \rightarrow \dots$$

Recall that when  $M$  and  $N$  are graded  $E$ -modules, their *graded* tensor product  $M \otimes_K^{\text{gr}} N$  and homomorphism space  $\text{Hom}_K^{\text{gr}}(N, M)$  have *diagonal actions*:

$$(x \otimes y)e_\sigma = \sum_{\tau \subseteq \sigma} (-1)^{k|\tau|} \text{sgn}_{\sigma \setminus \tau}^\tau x e_\tau \otimes y e_{\sigma \setminus \tau}$$

$$(\gamma e_\sigma)(y) = \sum_{\tau \subseteq \sigma} (-1)^{|\tau|(k+(|\tau|+1)/2)} \text{sgn}_{\sigma \setminus \tau}^\tau \gamma(y e_\tau) e_{\sigma \setminus \tau}$$

for  $y \in N_k$  and  $\sigma \subseteq [n]$

where  $\text{sgn}_{\sigma \setminus \tau}^\tau$  is the sign of the permutation  $(\tau, \sigma \setminus \tau)$ ; that these are (graded)  $E$ -modules follows from the fact that  $E$  is a *super* Hopf algebra.

The properties of  $V_E(M)$  are similar to those of the varieties of modular representations, but proofs are simpler, compare the account by Benson [5].

**Theorem 3.1.** *If the field  $K$  is algebraically closed, then the rank varieties of finite  $E$ -modules  $M, N$  satisfy the following properties.*

- (1)  $V_E(M)$  is a cone (that is, a homogeneous algebraic subset) in  $V$ .
- (2)  $\dim V_E(M) = \text{cx}_E M$  and  $2^{n - \text{cx}_E M}$  divides  $\text{rank}_K M$ .
- (3)  $V_E(M) = \{0\}$  if and only if  $M$  is free.
- (4)  $V_E(M) = V_E(N)$  if  $M$  is a syzygy of  $N$ .
- (5) If  $M \subseteq N$ , then each one of the three varieties  $V_E(M)$ ,  $V_E(N)$ ,  $V_E(N/M)$ , is contained in the union of the other two.
- (6)  $V_E(M \oplus N) = V_E(M) \cup V_E(N)$ .
- (7)  $V_E(M \otimes_K^{\text{gr}} N) = V_E(M) \cap V_E(N) = V_E(\text{Hom}_K^{\text{gr}}(N, M))$  if  $M, N$  are graded.
- (8) Each cone in  $V$  is the rank variety of some graded  $E$ -module.

As over commutative rings, the notion of regularity can be extended to sequences. Elements  $v_1, \dots, v_r \in V$  form an  $M$ -regular sequence if  $v_i$  is  $(M/M(v_1, \dots, v_{i-1}))$ -regular for  $1 \leq i \leq r$ , in other words, if  $yv_i \in M(v_1, \dots, v_{i-1})$  implies that  $y \in M(v_1, \dots, v_i)$  for  $1 \leq i \leq r$ . It is clear that each  $M$ -regular sequence can be extended to a maximal one. The supremum of the lengths of  $M$ -regular sequences is called the *depth* of  $M$  over  $E$ , and denoted  $\text{depth}_E M$ .

Parts of the preceding theorem depend on a depth-formula for modules over exterior algebras, that is similar to the extension of the classical Auslander-Buchsbaum equality to modules over complete intersections, obtained in [4].

**Theorem 3.2.** *If the field  $K$  is infinite and  $M$  is a finite  $E$ -module, then each maximal  $M$ -regular sequence has  $\text{depth}_E M$  elements, and*

$$\text{depth}_E M + \text{cx}_E M = n.$$

**Examples 3.3.** (1) If  $\text{rank}_K M$  is odd, then  $\text{cx}_E M = n$ .

Indeed, if  $\text{depth}_E M > 0$ , then taking an  $M$ -regular  $v \in V$  we get  $\text{rank}_K M = \text{rank}_K(\text{Ann}_M(v)) + \text{rank}_K(Mv) = 2 \text{rank}_K(Mv)$ , so  $\text{rank}_K M$  is even.

(2) The depth equality fails when  $K$  is finite and  $n \geq 2$ .

Indeed, if  $v \in V \setminus \{0\}$  then  $E \xrightarrow{\lambda_v} E \xrightarrow{\lambda_v} E$  with  $\lambda_v(e) = ve$  is an exact complex of  $E$ -modules, so  $\text{cx}_E(E/(v)) = 1$ , and hence  $M = \bigoplus_{v \in V} E/(v)$  has complexity 1; on the other hand, it is clear that  $V_E(M) = V$ , hence  $\text{depth}_R M = 0$ .

To start on the proofs, we record some simple facts on regularity.

**Remarks 3.4.** Let  $M$  be an  $E$ -module.

(1) When  $v^2 = 0$ , any  $K[v]$ -module is a direct sum of copies of  $K[v]$  and  $K[v]/(v)$ . Thus,  $v \in V = E_1$  is regular if and only if  $M$  is free over the subalgebra  $K[v] \subseteq E$ .

(2) For  $v \in V$ , let  $\pi: E \rightarrow E/(v)$  and  $\rho: M \rightarrow M/Mv$  be the canonical homomorphisms. If  $v$  is  $M$ -regular, then they induce isomorphisms

$$\begin{aligned} \text{Ext}_\pi^i(\rho, K): \text{Ext}_{E/(v)}^i(M/Mv, K) &\cong \text{Ext}_E^i(M, K) \\ \text{Tor}_\pi^i(\rho, K): \text{Tor}_i^E(M, K) &\cong \text{Tor}_i^{E/(v)}(M/Mv, K) \end{aligned} \quad \text{for } i \geq 0.$$

Indeed,  $M$  is free over  $K[v]$  by (2), so if  $G$  is a free resolution of  $M$  over  $E$ , then  $G/Gv$  is a free resolution of  $M/Mv$  over  $E/(v)$ . Thus,  $\text{Ext}_\pi^*(\rho, K)$  and  $\text{Tor}_\pi^*(\rho, K)$  are the maps induced in homology by the isomorphisms of complexes  $\text{Hom}_{E/(v)}(G/Gv, K) \cong \text{Hom}_E(G, K)$  and  $G \otimes_E K \cong (G/Gv) \otimes_{E/(v)} K$ , respectively.

(3) Regularity of a sequence  $\mathbf{v} = v_1, \dots, v_d \in V$  is detected by its *Cartan complex*  $C(\mathbf{v}; M)$ , defined by  $C_i(\mathbf{v}, M) = \bigoplus_{a \in \mathbb{N}^n, |a|=i} w^{(a)} M$  with  $w^{(a)} M \cong M$  for each  $a \in \mathbb{N}^n$  and  $\partial(w^{(a)} u) = \sum_{\ell \in \text{supp}(a)} w^{(a - \varepsilon_\ell)} u e_\ell$  for  $u \in M$ .

We set  $H(\mathbf{v}; M) = H(C(\mathbf{v}; M))$ , and note that the following are equivalent:

- (i)  $\mathbf{v}$  is  $M$ -regular.
- (ii)  $M$  is a free module over  $K[v_1, \dots, v_d]$ .
- (iii)  $H_1(\mathbf{v}; M) = 0$ .
- (iv)  $H_i(\mathbf{v}; M) = 0$  for  $i \geq 1$ .

Indeed, let  $E'$  be an exterior algebra on alternating variables  $e'_1, \dots, e'_d$ , and let  $\varphi: E' \rightarrow E$  be the homomorphism of  $K$ -algebras with  $\varphi(e'_i) = v_i$  for  $i = 1, \dots, d$ . If  $C'$  is the Cartan resolution of the right  $E'$ -module  $K$ , cf. Remark 2.5, then  $C(\mathbf{v}; M) = C' \otimes_E M$ , so  $H_i(\mathbf{v}; M) = \text{Tor}_i^{E'}(K, M)$ . Thus, (i)  $\implies$  (iv) by iterated use of (2). If (iii) holds, then  $\text{Tor}_1^{E'}(K, M) = 0$ ; computing Tor from a minimal free resolution of  $M$  over  $E'$  we see that  $M$  is free over  $E'$ ; it follows that  $\varphi$  is an isomorphism, so (iii)  $\implies$  (ii) holds. Finally, (ii)  $\implies$  (i) is trivial.

(4) By (3), each permutation of an  $M$ -regular sequence is itself  $M$ -regular.

To study the geometry of  $V_E(M)$  we use product structures in cohomology. We recall the basics, referring for details to MacLane [13] or Bourbaki [6].

**Construction 3.5.** For  $E$ -modules  $M, L, N$  and  $i, j \in \mathbb{Z}$ , *composition pairings*

$$\text{Ext}_E^j(L, N) \times \text{Ext}_E^i(M, L) \rightarrow \text{Ext}_E^{i+j}(M, N)$$

are introduced as follows. Let  $C$  and  $G$  be  $E$ -free resolutions of  $L$  and  $M$ , respectively, and represent elements in  $\text{Ext}_E^i(M, L)$  and  $\text{Ext}_E^j(L, N)$  by  $E$ -linear homomorphisms  $\varkappa: G_i \rightarrow L$  with  $\varkappa \partial_{i+1} = 0$  and  $\xi: C_j \rightarrow N$  with  $\xi \partial_{j+1} = 0$ . Choosing a lifting of  $\varkappa$  to an  $E$ -linear chain map  $\tilde{\varkappa}: G \rightarrow C$  of degree  $-i$ , define the product  $\text{cl}(\xi) \text{cl}(\varkappa)$  to be the class of the composition  $\xi \tilde{\varkappa}_{i+j}: G_{i+j} \rightarrow N$ .

The pairings are  $K$ -bilinear, associative, and natural (hence, independent of the choices made above). They make  $\text{Ext}_E^*(K, K) = \bigoplus_{i=0}^\infty \text{Ext}_E^i(K, K)$  into a graded algebra, and  $\text{Ext}_E^*(M, K) = \bigoplus_{i=0}^\infty \text{Ext}_E^i(M, K)$  into a graded left module over it.

**Proposition 3.6.** *There is a natural in  $V$  isomorphism of graded  $K$ -algebras*

$$\text{Ext}_E^*(K, K) \cong \text{Sym}_K^*(V^\vee) \quad \text{where } V^\vee = \text{Hom}_K(V, K).$$

*If  $M$  is a finite  $E$ -module, then the  $\text{Ext}_E^*(K, K)$ -module  $\text{Ext}_E^*(M, K)$  is finite.*

*Proof.* Cartan's resolution  $(C, \partial)$  of  $K$  over  $E$ , cf. Example 2.5, is minimal, so

$$\mathrm{Ext}^i(K, K) = \mathrm{H}^i(\mathrm{Hom}_E(C, K)) = \mathrm{Hom}_E\left(\bigoplus_{a \in \mathbb{N}^n, |a|=i} Ew^{(a)}, K\right).$$

The homomorphisms of  $E$ -modules  $\{\chi^a: C_i \rightarrow K \mid a \in \mathbb{N}^n, |a| = i\}$ , such that  $\chi^a(w^{(b)}) = 1$  for  $b = a$  and  $\chi^a(w^{(b)}) = 0$  for  $b \in \mathbb{N}^n$  with  $|b| = i$  and  $b \neq a$  form a  $K$ -basis of  $\mathrm{Hom}_E(C, K)$ . The  $E$ -linear maps

$$\tilde{\chi}_{i+j}^a: C_{i+j} \rightarrow C_j \quad \text{defined by} \quad \tilde{\chi}_{i+j}^a(w^{(b)}) = \begin{cases} w^{(b-a)} & \text{if } b-a \in \mathbb{N}^n; \\ 0 & \text{otherwise,} \end{cases}$$

define a lifting of  $\chi^a$  to a chain map  $C \rightarrow C$ . This means that  $\chi^a \chi^b = \chi^{a+b}$  for all  $b \in \mathbb{N}^n$ , so  $\mathrm{Ext}_E^*(K, K)$  is the polynomial ring on  $\chi_1 = \chi^{e_1}, \dots, \chi_n = \chi^{e_n}$ .

To see that the  $\mathrm{Ext}_E^*(K, K)$ -module  $\mathrm{Ext}_E^*(M, K)$  is finite we argue by induction on  $q = \max\{r \mid ME_r \neq 0\}$ . If  $q = 1$ , then  $M \cong K^s$  for some  $s$  and the assertion is clear. If  $q > 1$ , then  $M' = M(V) \neq 0$ , so the exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of  $E$ -modules yields an exact sequence of  $\mathrm{Ext}_E^*(K, K)$ -modules

$$\mathrm{Ext}_E^*(M', K) \rightarrow \mathrm{Ext}_E^*(M, K) \rightarrow \mathrm{Ext}_E^*(M'', K) \quad (3.6.1)$$

in which those on the outside are noetherian by the induction hypothesis.  $\square$

**Remark 3.7.** If  $\chi_1, \dots, \chi_n$  is the basis of  $V^\vee$  dual to the basis  $e_1, \dots, e_n$  of  $V$ , then we identify  $\mathrm{Ext}_E^*(K, K)$  with the graded polynomial ring  $\mathcal{S} = K[\chi_1, \dots, \chi_n]$  in which each  $\chi_i$  has degree 1; the elements of  $\mathcal{S}$  act as functions on  $V$ .

Applied to the  $\mathcal{S}$ -module  $\mathrm{Ext}_E^*(M, K)$ , the Hilbert-Serre theorem yields:

**Corollary 3.8.** *The Krull dimension of the  $\mathcal{S}$ -module  $\mathrm{Ext}_E^*(M, K)$  is equal to  $\mathrm{cx}_E M$ , and there exists a polynomial  $p_M(t) \in \mathbb{Z}[t]$  with  $p_M(1) > 0$ , such that*

$$P_M^E(t) = \frac{p_M(t)}{(1-t)^c} \quad \text{with} \quad c = \mathrm{cx}_E M. \quad \square$$

Now we give a basic cohomological description of the rank variety.

**Theorem 3.9.** *If  $K$  is algebraically closed and  $M$  is a finite  $E$ -module, then*

$$V_E(M) = \{v \in V \mid \xi(v) = 0 \text{ for all } \xi \in \mathrm{Ann}_{\mathcal{S}}(\mathrm{Ext}_E^*(M, K))\}.$$

*Proof.* Let  $\mathcal{I} = \mathrm{Ann}_{\mathcal{S}}(\mathrm{Ext}_E^*(M, K))$ . For  $v \in V$ , set  $V^v = \mathrm{Ker}(V^\vee \rightarrow (vK)^\vee)$ , and let  $\mathcal{P}_v$  denote the homogeneous prime ideal  $(V^v)$  of  $\mathcal{S}$ . By the Nullstellensatz, we have to prove that  $\mathcal{I} \subseteq \mathcal{P}_v$  if and only if  $v$  is  $M$ -singular.

If  $v$  is singular, then by Remark 3.4.1 we have an isomorphism of  $K[v]$ -modules  $M \cong K[v]^p \oplus K^q$  with  $q > 0$ . The inclusion  $\iota: K[v] \hookrightarrow E$  induces a diagram

$$\begin{array}{ccc} \mathrm{Ext}_E^*(K, K) \otimes_K \mathrm{Ext}_E^*(M, K) & \longrightarrow & \mathrm{Ext}_E^*(M, K) \\ \mathrm{Ext}_i^*(K, K) \otimes \mathrm{Ext}_i^*(M, K) \downarrow & & \downarrow \mathrm{Ext}_i^*(M, K) \\ \mathrm{Ext}_{K[v]}^*(K, K) \otimes_K \mathrm{Ext}_{K[v]}^*(M, K) & \longrightarrow & \mathrm{Ext}_{K[v]}^*(M, K). \end{array}$$

It commutes by naturality of composition products, so  $\mathrm{Ext}_i^*(K, K)(\mathcal{I})$  annihilates

$$\mathrm{Ext}_{K[v]}^*(M, K) \cong K^p \oplus \mathrm{Ext}_{K[v]}^*(K, K)^q.$$

It is then equal to 0, that is,  $\mathcal{I} \subseteq \mathrm{Ker} \mathrm{Ext}_i^*(K, K) = \mathcal{P}_v$ .

If  $v$  is regular, then  $\pi: E \rightarrow E/(v)$  and  $\rho: M \rightarrow M/Mv$  induce a diagram

$$\begin{array}{ccc} \text{Ext}_E^*(K, K) \otimes_K \text{Ext}_E^*(M, K) & \longrightarrow & \text{Ext}_E^*(M, K) \\ \text{Ext}_\pi^*(K, K) \otimes \text{Ext}_\pi^*(\rho, K) \uparrow & & \uparrow \text{Ext}_\pi^*(\rho, K) \\ \text{Ext}_{E/(v)}^*(K, K) \otimes_K \text{Ext}_{E/(v)}^*(M/Mv, K) & \longrightarrow & \text{Ext}_{E/(v)}^*(M/Mv, K). \end{array}$$

It is commutative by naturality, and  $\text{Ext}_\pi^*(\rho, K)$  is an isomorphism by Remark 3.4.2. As  $\text{Ext}_{E/(v)}^*(M/Mv, K)$  is a finite  $\text{Ext}_{E/(v)}^*(K, K)$ -module by Proposition 3.6, we conclude that so is  $\text{Ext}_E^*(M, K)$ . It follows that the composition

$$\text{Ext}_{E/(v)}^*(K, K) \xrightarrow{\text{Ext}_\pi^*(K, K)} \text{Ext}_E^*(K, K) = \mathcal{S} \rightarrow \mathcal{S}/\mathcal{I}$$

is a finite homomorphism of rings. Assuming that  $\mathcal{I} \subseteq \mathcal{P}_v$ , we conclude that

$$\text{Sym}_K^*[V^v] \cong \text{Ext}_{E/(v)}^*(K, K) \rightarrow \mathcal{S}/\mathcal{P}_v = \text{Ext}_{K[v]}^*(K, K) \cong \text{Sym}_K^*[(Kv)^\vee]$$

is a finite homomorphism: this is absurd, since it maps  $V^v$  to 0.  $\square$

*Proof of Theorem 3.2.* Let  $\mathbf{v} = v_1, \dots, v_d$  be an arbitrary maximal  $M$ -regular sequence in  $V$ . We want to prove that  $\text{depth}_E M = d$  and  $\text{cx}_E M = n - d$ .

We first assume that  $K$  is algebraically closed; the elements in a regular sequence being  $K$ -linearly independent, we have  $d \leq n$ , so we can induce on  $d$ . An equality  $d = 0$  means that each element of  $V$  is  $M$ -singular, that is,  $\text{depth}_E M = 0$ ; on the other hand, Theorem 3.9 yields  $\text{cx}_R M = \dim V_E(M) = \dim V = n$ .

If  $d > 0$ , then the images of  $v_2, \dots, v_d$  in  $E/(v_1)$  form a maximal  $(M/Mv_1)$ -regular sequence. The induction hypothesis yields  $\text{depth}_E(M/Mv_1) = d - 1$  and

$$\text{cx}_{E/(v_1)}(M/Mv_1) = (n - 1) - (d - 1) = n - d.$$

As  $\text{cx}_{E/(v_1)}(M/Mv_1) = \text{cx}_E M$  by Remark 3.4.2, we are done.

Let now  $K$  be an arbitrary infinite field. Taking an algebraic closure  $\bar{K}$  of  $K$ , we consider the finite module  $\bar{M} = M \otimes_K \bar{K}$  over the exterior algebra  $\bar{E} = E \otimes_K \bar{K}$  of the  $\bar{K}$ -vector space  $\bar{V} = V \otimes_K \bar{K}$ . Due to the flatness of  $\bar{E}$  over  $E$ , we see that (considered as a sequence in  $\bar{V}$ ) any  $M$ -regular sequence in  $V$  is  $\bar{M}$ -regular, and that  $\beta_i^{\bar{E}}(\bar{M}) = \beta_i^E(M)$  for each  $i$ . This yields

$$\text{depth}_E M \leq \text{depth}_{\bar{E}} \bar{M} = d \quad \text{and} \quad \text{cx}_E M = \text{cx}_{\bar{E}} \bar{M} = n - d.$$

Assuming that the  $\bar{M}$ -regular sequence  $\mathbf{v}$  is not maximal, we can find in  $\bar{V}/\bar{K}\mathbf{v}$  an element  $v$  that is  $(\bar{M}/\bar{M}(\mathbf{v}))$ -regular. As the set of regular elements is Zariski-open and  $K$  is infinite, we can even pick  $v$  in  $V/(\mathbf{v})$ , and get an  $M$ -regular sequence  $\mathbf{v}, v$ . This is absurd, so  $\mathbf{v}$  is a maximal  $\bar{M}$ -regular sequence and we have

$$d \leq \text{depth}_E M \leq \text{depth}_{\bar{E}} \bar{M} = d.$$

It follows that  $\text{depth}_E M = d$  and  $\text{depth}_E M + \text{cx}_E M = n$ , as desired.  $\square$

**Lemma 3.10.** *For each  $\xi \in \text{Ext}_E^i(K, K)$  there is a graded  $E$ -module  $L_\xi$  such that*

$$V_E(L_\xi) = \{v \in V \mid \xi(v) = 0\}.$$

*Proof.* In the Cartan resolution  $C$  of  $K$  over  $E$ , set  $D_i = \partial_i(C_i)$ , let  $\bar{\xi}: D_i \rightarrow K$  be the  $E$ -linear map that corresponds to  $\xi$  under the isomorphisms

$$\text{Ext}^i(K, K) = \text{Hom}_E(C_i, K) \cong \text{Hom}_E(D_i, K)$$

and set  $L_\xi = \text{Ker } \bar{\xi}$ . The exact sequence of  $E$ -modules

$$0 \rightarrow L_\xi \rightarrow D_i \rightarrow K \rightarrow 0$$

induces an exact sequence of graded modules over  $\mathcal{S} = \text{Ext}^*(K, K)$ ,

$$\mathcal{S} \xrightarrow{\bar{\xi}^*} \text{Ext}_E^*(D_i, K) \rightarrow \text{Ext}_E^*(L_\xi, K) \xrightarrow{\bar{\sigma}} \mathcal{S}(1) \xrightarrow{\bar{\xi}^*(1)} \text{Ext}_E^*(D_i, K)(1)$$

where  $\bar{\xi}^* = \text{Ext}_E^*(\bar{\xi}, K)$  maps  $1 \in \mathcal{S}^0$  to  $\xi \in \text{Ext}_E^i(D_i, K) = \mathcal{S}^i$ . Thus,  $\bar{\xi}^*$  and  $\bar{\xi}^*(1)$  are injective, yielding  $\text{Ext}_E^*(L_\xi, K) \cong \mathcal{S}^{\geq i}(i)/\mathcal{S}\xi$ . As  $\sqrt{\mathcal{S}^{\geq i}(i)/\mathcal{S}\xi} = \sqrt{\mathcal{S}\xi}$ , we conclude from Theorem 3.9 that  $V_E(L_\xi)$  has the desired form.  $\square$

*Proof of Theorem 3.1.* (1) Note that  $\text{rank}_K(Mv) \leq \text{rank}_K(\text{Ann}_M(v))$  for each  $v \in V$ , and the inequality is strict precisely when  $v$  is  $M$ -singular. Setting  $m = \text{rank}_K M$ , we rewrite the inequality as  $\text{rank}_K(\rho^v) < m - \text{rank}_K(\rho^v)$ , that is, as  $\text{rank}_K(\rho^v) < m/2$ . Thus,  $V_E(M)$  is the zero-set of the minors of order  $\lceil m/2 \rceil$  of a matrix representing multiplication by a generic element of  $V$ . Clearly,  $v \in V_E(M)$  implies  $\lambda v \in V_E(M)$  for each  $\lambda \in K$ , so the variety is homogeneous.

(2) Let  $\text{cx}_E M = c$ . By Corollary 3.8 and elementary dimension theory, the number  $c$  is equal to the Krull dimension of the ring  $\mathcal{S}/\text{Ann}_{\mathcal{S}}(\text{Ext}_E^*(M, K))$ , which is the dimension of the variety  $V_E(M)$ .

Theorem 3.2 yields an  $M$ -regular sequence  $v_1, \dots, v_{n-c}$  in  $V$ , so  $M$  is free over  $E' = K[v_1, \dots, v_{n-c}]$  by Remark 3.4, so  $\text{rank}_K M = 2^{n-c} \text{rank}_{E'} M$ .

(3) If  $V_E(M) = \{0\}$ , then  $\text{cx}_E M = 0$ , so the preceding argument works with  $r = n$ , and shows that  $M$  is free over  $K[v_1, \dots, v_n] = E$ . Conversely, if  $M$  is free over  $E$  the non-zero elements of  $V$  are obviously  $M$ -regular, hence  $V_E(M) = \{0\}$ .

(5) An exact sequence of  $E$ -modules  $0 \rightarrow M \rightarrow N \rightarrow M/N \rightarrow 0$  induces an exact sequence of complexes of vector spaces

$$0 \rightarrow (M, \rho^v) \rightarrow (N, \rho^v) \rightarrow (M/N, \rho^v) \rightarrow 0$$

and hence an exact sequence of homology spaces

$$\mathbb{H}_*(M, \rho^v) \rightarrow \mathbb{H}_*(N, \rho^v) \rightarrow \mathbb{H}_*(M/N, \rho^v) \rightarrow \mathbb{H}_*(M, \rho^v) \rightarrow \mathbb{H}_*(N, \rho^v)$$

which implies the desired assertions follow immediately.

(4) It suffices to consider the case when  $M$  and  $N$  appear in an exact sequence  $0 \rightarrow M \rightarrow P \rightarrow N \rightarrow 0$  with a free  $E$ -module  $P$ . By (5) and (3) we then have

$$V_E(M) \subseteq V_E(N) \cup V_E(P) = V_E(N) \subseteq V_E(M) \cup V_E(P) = V_E(M).$$

(6) follows immediately from the definitions.

(7) Recall that  $v \in V$  acts on  $M \otimes_K^{\text{gr}} M$  by the formula  $(x \otimes y)v = x \otimes yv + (-1)^k xv \otimes y$ , when  $y \in N_k$ . This means that  $x \otimes y \mapsto y \otimes x$  is an isomorphism

$$(M \otimes_K^{\text{gr}} N, v) \cong (N, v) \otimes_K (M, v)$$

where the tensor product on the right hand side is one of complexes of  $K$ -vector spaces. The Künneth formula then gives an isomorphism of graded vector spaces

$$\mathbb{H}^*(M \otimes_K^{\text{gr}} N, v) \cong \mathbb{H}^*(N, v) \otimes_K \mathbb{H}^*(M, v)$$

from which we get  $V_E(M \otimes_K^{\text{gr}} N) = V_E(M) \cap V_E(N)$ .

A similar argument yields  $\mathbb{H}^*(\text{Hom}_K^{\text{gr}}(N, M), v) \cong \text{Hom}_K(\mathbb{H}^*(N, v), \mathbb{H}^*(M, v))$ , establishing the equality  $V_E(\text{Hom}_K^{\text{gr}}(N, M)) = V_E(M) \cap V_E(N)$ .

(8) Given a cone  $W \subseteq V$ , pick homogeneous polynomials  $\xi_1, \dots, \xi_s \in \mathcal{S}$  that define it, and note that  $W = V_E(L_{\xi_1} \otimes_K^{\text{gr}} \dots \otimes_K^{\text{gr}} L_{\xi_s})$  by (7) and Lemma 3.10.  $\square$

## 4. SIMPLICIAL COMPLEXES

For  $\sigma \subseteq [n]$ , let  $K\sigma$  denote the coordinate subspace spanned by  $\{e_j \mid j \in \sigma\}$ .

In a  $n$ -graded situation, we refine some results of the preceding section as follows.

**Proposition 4.1.** *Let  $M$  be a finite  $n$ -graded  $E$ -module.*

- (1)  $\text{Ext}_E^*(M, K)$  is a finite  $(1+n)$ -graded left module over the polynomial ring  $\mathcal{S} = K[\chi_1, \dots, \chi_n]$ , in which  $\chi_i$  has  $(1+n)$ -degree  $(1, \varepsilon_i)$ .
- (2) There exists a polynomial  $p_M(t, u_1, \dots, u_n) \in \mathbb{Z}[t, u_1, \dots, u_n]$  such that

$$P_M^E(t, u_1, \dots, u_n) = \frac{p_M(t, u_1, \dots, u_n)}{\prod_{j=1}^n (1 - tu_j)};$$

if  $M_a = 0$ , then no monomial  $t^i u^a$  appears in  $p_M(t, u_1, \dots, u_n)$ .

- (3) The variety  $V_E(M)$  is a union of coordinate subspaces of  $V$ .
- (4) Each union of coordinate subspaces is the variety of a  $n$ -graded  $E$ -module.

*Proof.* (1) Take a  $n$ -graded free resolution  $G$  of  $M$ , and let  $\text{Ext}_E^{ia}(M, K)$  consist of those elements of  $\text{Ext}_E^i(M, K) = H^i \text{Hom}(G, K)$  that can be represented by a homomorphism  $\varkappa: G_i \rightarrow K$ , such that  $\varkappa(G_{ib}) = 0$  when  $a \neq b \in \mathbb{Z}^n$ . Performing Construction 3.5 with this  $G$  and the  $n$ -graded Cartan resolution  $C$  of  $K$ , cf. Example 2.5, and using  $n$ -homogeneous maps, one gets bilinear pairings

$$\text{Ext}_E^{jb}(K, K) \times \text{Ext}_E^{ia}(M, K) \rightarrow \text{Ext}_E^{i+j, a+b}(M, K) \quad \text{for all } i, j \in \mathbb{Z}; a, b \in \mathbb{Z}^n.$$

They make  $\text{Ext}_E^*(M, K)$  into a  $(1+n)$ -graded left module over  $\text{Ext}_E^*(K, K)$ , and the identification  $\text{Ext}_E^*(K, K) = \mathcal{S}$  of Remark 3.7 is compatible with this grading.

(2) The expression for  $P_M^R(t, u_1, \dots, u_n)$  comes from (1), by the multigraded version of the Hilbert-Serre theorem. The assertion on the monomials in the numerator is obvious when  $M \cong \bigoplus_{i=1}^s K(a_i)$  with  $a_i \in \mathbb{Z}^n$ . As (3.6.1) is an exact sequence of  $(1+n)$ -graded vector spaces, we conclude by induction on  $\text{rank}_K M$ .

(3) The annihilator of the multigraded  $\mathcal{S}$ -module  $\text{Ext}_E^*(M, K)$  being a monomial ideal in  $\chi_1, \dots, \chi_n$ , its radical is an intersection of prime ideals generated by subsets of  $\{\chi_1, \dots, \chi_n\}$ . The desired assertion follows from Theorem 3.9.

(4) Note that  $\bigcap_{i=1}^s V_E(K\sigma_i) = V_E(\bigoplus_{i=1}^s E/(K\sigma_i))$ . □

**Theorem 4.2.** *If  $J$  is a monomial ideal in  $E$  and  $I$  is the corresponding squarefree monomial ideal in  $\mathcal{S}$ , then*

$$V_E(E/J) = \bigcup_{a \in \Sigma} K \text{supp}(a)$$

where  $\Sigma$  is the set of shifts of a minimal free resolution of  $\mathcal{S}/I$  over  $\mathcal{S}$ , and so

$$\text{cx}_E(E/J) = \max\{|a| \mid a \in \Sigma\}.$$

The proof of the theorem is deferred to the end of the section.

Let  $\Delta$  be a simplicial complex with  $n$  vertices, and set  $K\langle\Delta\rangle = E/J$ , where  $J$  is generated by  $\{e_\sigma \mid \sigma \notin \Delta\}$ . We give a combinatorial interpretation of the complex

$$(K\langle\Delta\rangle, v): \quad 0 \rightarrow K\langle\Delta\rangle_1 \xrightarrow{\rho^v} K\langle\Delta\rangle_2 \xrightarrow{\rho^v} \dots$$

For a subset  $\rho \subseteq [n]$ , we denote  $\Delta_\rho$  the restriction of  $\Delta$  to  $\rho$ , that is, the simplicial complex with faces  $\sigma \in \Delta$  such that  $\sigma \subseteq \rho$ . Furthermore, for a face  $\sigma \in \Delta$  we introduce the *link of  $\sigma$  in  $\Delta_\rho$*  as the simplicial complex

$$\text{lk}_{\Delta_\rho} \sigma = \langle \tau \in \Delta_\rho \mid \tau \cup \sigma \in \Delta \rangle.$$

For  $v \in V$ ,  $v = \sum_{i=1}^n \lambda_i e_i$ , we call  $\text{supp}(v) = \{i \mid \lambda_i \neq 0\}$  the *support* of  $v$ .

Now the cohomology of  $(K\langle\Delta\rangle, v)$  can be interpreted as follows:

**Proposition 4.3.** *The complex  $(K\langle\Delta\rangle, v)$  only depends on  $\rho = \text{supp}(v)$ , namely, it is isomorphic to  $(K\langle\Delta\rangle, v_\rho)$  with  $v_\rho = \sum_{j \in \rho} e_j$ . Furthermore,*

$$H^i(K\langle\Delta\rangle, v) \cong \bigoplus_{\sigma \in \Delta, \sigma \subseteq [n] \setminus \rho} \tilde{H}^{i-1}(\text{lk}_{\Delta_\rho} \sigma; K)$$

where  $\tilde{H}^*(; K)$  denotes reduced simplicial cohomology with coefficients in  $K$ .

*Proof.* The map  $\varphi: V \rightarrow V$  given by  $\varphi(e_j) = \lambda_j^{-1} e_j$  for  $j \in \rho$  and  $\varphi(e_j) = e_j$  for  $j \notin \rho$  extends to an isomorphism of  $K$ -algebras  $\varphi: K\langle\Delta\rangle \rightarrow K\langle\Delta\rangle$ , with  $\varphi(v) = v_\rho$ .

As a  $K\langle\Delta_\rho\rangle$ -module the algebra  $K\langle\Delta\rangle$  decomposes as follows:

$$K\langle\Delta\rangle = \bigoplus_{\sigma \in \Delta, \sigma \subseteq [n] \setminus \rho} e_\sigma \cdot K\langle\Delta_\rho\rangle.$$

Now note that  $e_\sigma K\langle\Delta_\rho\rangle \cong K\langle\text{lk}_{\Delta_\rho} \sigma\rangle$ , and that  $(K\langle\text{lk}_{\Delta_\rho} \sigma\rangle, v)$  is isomorphic to the augmented oriented cochain complex of  $\text{lk}_{\Delta_\rho} \sigma$  with values in  $K$ .  $\square$

By a theorem of Hochster [12],  $\rho \subseteq [n]$  is the support of a shift of the resolution of  $k[\Delta]$  if and only if  $\tilde{H}(\Delta_\rho; K) \neq 0$ , so Theorem 4.2 and Proposition 4.3 yield

**Corollary 4.4.** *Let  $\Delta$  be a simplicial complex with  $n$  vertices. For a subset  $\sigma \subseteq [n]$  and a field  $K$  the following conditions are equivalent:*

- (i) *There exists  $\rho \subseteq [n]$  with  $\sigma \subseteq \rho$  such that  $\tilde{H}(\Delta_\rho; K) \neq 0$ .*
- (ii) *There exists  $\tau \in \Delta$  with  $\tau \cap \sigma = \emptyset$ , such that  $\tilde{H}(\text{lk}_{\Delta_\tau} \sigma; K) \neq 0$ .*  $\square$

We single out a special case: For any simplicial complex  $\Delta$  with  $\tilde{H}^*(\Delta; k) \neq 0$  and any subset  $\sigma$  of the vertex set of  $\Delta$ , there is a face  $\tau$  of  $\Delta$  such that  $\tilde{H}(\text{lk}_{\Delta_\tau} \sigma; K) \neq 0$ .

*Proof of Theorem 4.2.* Let  $F$  be a minimal free resolution of  $S/I$  over  $S$ , let  $G$  be the minimal free resolution of  $E/J$  over  $E$  of Theorem 1.3, and let  $Y_\ell$  be the basis of  $G_\ell$  from Construction 1.1. A homogeneous  $K$ -basis of  $\text{Hom}_E(G_\ell, K) = \text{Ext}_E^\ell(E/J, K)$  is given by  $\{\varkappa_f^a \mid \varkappa_f^a(y^{(a)} f) = 1 \text{ and } \varkappa_f^a(Y_\ell \setminus \{y^{(a)} f\}) = 0\}$ .

In the Cartan resolution  $C$  of  $K$  over  $E$ , cf. Example 2.5, set  $1 = w^{(0)}$  and  $w_j = w^{(\varepsilon_j)}$ . Fixing a homomorphisms  $\varkappa_f^a: G_\ell \rightarrow K$ , with  $f \in B_i$  and  $\deg(f) = b$ , we note that a lifting of  $\varkappa_f^a$  to a chain map  $\tilde{\varkappa}_f^a: G \rightarrow C$  can be started by

$$\begin{aligned} (\tilde{\varkappa}_f^a)_\ell(y^{(a')} f') &= \begin{cases} 1 & \text{when } a = a' \text{ and } f = f'; \\ 0 & \text{otherwise;} \end{cases} \\ (\tilde{\varkappa}_f^a)_{\ell+1}(y^{(a')} f') &= \begin{cases} (-1)^{|b|} w_j & \text{when } a = a' + \varepsilon_j, j \in \text{supp}(b), \text{ and } f = f'; \\ & \text{when } a = a', j \in \text{supp}(b' - b), \\ & \text{and } \theta(f') = \sum_{g \in B_i} \lambda_{f'g} x^{b'-c} g \\ & \text{with } b' = \deg(f'), c = \deg(g); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

These cases are disjoint because  $b'$  is squarefree, so by Construction 3.5 we have

$$\chi_j \varkappa_f^a = \begin{cases} (-1)^{|b|} \varkappa_f^{a+\varepsilon_j} & \text{for } j \in \text{supp}(f); \\ (-1)^{|a|} \sum_{f' \in B_{i+1}: b'=b+\varepsilon_j} \lambda_{f'f} \varkappa_{f'}^a, & \text{for } j \in \text{null}(f) = [n] \setminus \text{supp}(f). \end{cases}$$

Ordering the subsets of  $[n]$  by inclusion, we set  $B[0] = \emptyset$  and

$$B[p] = \{ f \in B \setminus B[p-1] \mid \text{supp}(f) \text{ is maximal in } B \setminus B[p-1] \} \quad \text{for } p \geq 1.$$

The multiplication table shows that the  $K$ -span of  $\{ \varkappa_f^q \mid \text{supp}(f) \in \bigcup_{p \leq q} B[p] \}$  is a submodule  $\mathcal{M}[q]$  of  $\mathcal{M} = \text{Ext}_E^*(M, K)$  over  $\mathcal{S} = K[\chi_1, \dots, \chi_n]$ , such that

$$\frac{\mathcal{M}[q]}{\mathcal{M}[q-1]} \cong \bigoplus_{f \in B[q]} \mathcal{S} \varkappa_f^0 \quad \text{and} \quad \text{Ann}_{\mathcal{S}}(\varkappa_f^0) = (\text{null}(f)).$$

From the finite filtration  $0 = \mathcal{M}[0] \subseteq \dots \subseteq \mathcal{M}[n] = \mathcal{M}$  we then get

$$\sqrt{\text{Ann}_{\mathcal{S}} \mathcal{M}} = \sqrt{\bigcap_{q=1}^n \text{Ann}_{\mathcal{S}} \frac{\mathcal{M}[q]}{\mathcal{M}[q-1]}} = \bigcap_{q=1}^n \sqrt{\text{Ann}_{\mathcal{S}} \frac{\mathcal{M}[q]}{\mathcal{M}[q-1]}} = \bigcap_{f \in B} (\text{null}(f)).$$

The desired result now follows from Theorem 3.9.  $\square$

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