

# LAURENT COEFFICIENTS AND EXT OF FINITE GRADED MODULES

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ABSTRACT. Let  $M$  and  $N$  be finite graded modules over a graded commutative ring generated over a field  $K = R_0$  by homogeneous elements  $x_1, \dots, x_e$  of positive degrees  $d_1, \dots, d_e$ . By the Hilbert-Serre Theorem, the Hilbert series  $\sum_{n \in \mathbb{Z}} (\text{rank}_K M_n) t^n$  is the Laurent expansion around  $t = 0$  of a rational function  $H_M(t) = q_M(t) / \prod_{i=1}^e (1 - t^{d_i})$  with  $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ . The main result in this paper establishes an equality of rational functions

$$\sum_i (-1)^i H_{\text{Ext}_R^i(M, N)}(t) = \frac{H_M(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})}$$

when  $\text{Ext}_R^i(M, N) = 0$  for  $i \gg 0$ . No finiteness assumption is imposed on the resolution of either module.

Two applications are given. The first produces nontrivial lower bounds for the Bass numbers of a finite graded module of finite injective dimension over a graded algebra generated by elements of degree 1. The second relates the coefficients of the Laurent expansions of  $H_M(t)$  and  $H_N(t)$  around  $t = 1$  to certain alternating sums of Laurent coefficients of their Ext modules. Among the formulas which arise in this way is one recently discovered by Benson and Crawley-Boevey [*A ramification formula for Poincaré series, and a hyperplane formula for modular invariants*, Bull. London Math. Soc. **27** (1995), 435–440].

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Let  $R = \bigoplus_{n \geq 0} R_n$  be a graded commutative ring generated over a field  $K = R_0$  by homogeneous elements  $x_1, \dots, x_e$  of positive degrees  $d_1, \dots, d_e$ . The Hilbert-Serre Theorem shows that for each finite graded  $R$ -module  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  the *Hilbert series*  $\sum_{n \in \mathbb{Z}} (\text{rank}_K M_n) t^n$  is the Laurent expansion around  $t = 0$  of a rational function

$$H_M(t) = \frac{q_M(t)}{\prod_{i=1}^e (1 - t^{d_i})}$$

with  $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ . Laurent expansions  $[M]_z$  of  $H_M(t)$  around other points  $z$  of the extended complex plane  $\overline{\mathbb{C}}$  also carry important structural information.

For instance, when  $R$  is generated in degree one, the principal part of  $[M]_1$  determines the Hilbert polynomial of  $M$ . Without assumptions on the generation of  $R$ , coefficients of  $[M]_1$  have been sighted in invariant theory. In that context Benson and Crawley-Boevey [4], cf. also [3], have discovered recently that for all finite graded modules over a graded normal domain the coefficient  $\psi$  of  $1/(1-t)^{\dim R-1}$  satisfies

$$(1) \quad \begin{aligned} & \psi(\text{Hom}_R(M, N)) - \psi(\text{Ext}_R^1(M, N)) = \\ & \text{rank}_R(M) \text{rank}_R(N) \psi(R) + \text{rank}_R(M) \psi(N) - \text{rank}_R(N) \psi(M). \end{aligned}$$

The present investigation started as an attempt to “explain” this intriguing formula, and to find out whether analogous equalities exist for subsequent coefficients of Laurent expansions around 1. Our approach is to embed (1) into an infinite sequence of similar relations, derived by comparing the coefficients of Laurent expansions around 1 of the rational functions  $\sum_{i=0}^j (-1)^i H_{\text{Ext}_R^i(M, N)}(t)$  for  $j = 0, 1, \dots$  with those of

$$\phi_R(M, N)(t) = \frac{H_M(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})}.$$

In Section 3 we prove that the zeroth predicted relation holds over any domain, that the first one coincides with (1) and holds over domains which are regular in codimension 1, and that the second one holds over factorial domains which are regular in codimension 2. However, normality does not suffice for the validity of the second relation, as shown by examples in Section 4.

Our main result establishes an equality of rational functions  $\sum_i (-1)^i H_{\text{Ext}_R^i(M, N)}(t) = \phi_R(M, N)(t)$ , whenever  $\text{Ext}_R^i(M, N) = 0$  for  $i \gg 0$ . It is virtually obvious when  $M$  has a finite free resolution  $F^\bullet$ : In the Grothendieck group of  $R$ , the alternating sum of the homologies of the finite complex  $\text{Hom}_R(F^\bullet, N)$  is the same as the alternating sum of the terms of the complex. The result established here makes no finiteness assumption on the resolution of either  $M$  or  $N$ ; the proof uses permanence of Euler characteristics in spectral sequences.

Attempting to eliminate the finiteness assumption also on the cohomologies, one needs to attach a meaning to an infinite alternating sum of Hilbert functions. This may be achieved through an embedding of  $\mathbb{C}(t)$  into some topological field, and expansion into

Laurent series at some point  $z \in \overline{\mathbb{C}}$  is the most natural choice. However,  $z = 0$  is ruled out: convergence in  $\mathbb{C}((t))$  requires the order of the pole of  $\sum_{i=0}^j (-1)^i [\text{Ext}_R^i(M, N)]_0$  to *increase* with  $j$ , whereas for  $N$  of finite length these orders *decrease* with  $j$ . This prompts the use of Laurent expansions around  $z = \infty$  instead, and then indeed the alternating sum converges to  $[\phi_R(M, N)(t)]_\infty$  provided the length of  $N$  is finite. The question remains whether expansion around infinity will always work.

As a byproduct of our main result, we establish in Section 2 lower bounds for the Bass numbers of certain graded modules of finite injective dimension. They are analogous to known lower bounds [1] on the Betti numbers of modules of finite projective dimension.

### 1. HILBERT SERIES

When  $M$  and  $N$  are graded  $R$ -modules,  $\text{Hom}_R(M, N)_n$  stands for the  $K$ -vector subspace of  $\text{Hom}_R(M, N)$  consisting of homomorphisms  $\alpha$  of *degree*  $n$ , that is, such that  $\alpha(M_j) \subseteq N_{n+j}$  for  $j \in \mathbb{Z}$ . For a finite  $M$  the inclusion  $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_R(M, N)_n \subseteq \text{Hom}_R(M, N)$  is an equality, and thus  $\text{Hom}_R(M, N)$  is graded; as  $M$  has a resolution by finite free graded  $R$ -modules and homomorphisms of degree 0, this induces for each  $i$  a grading on  $\text{Ext}_R^i(M, N)$ .

**Theorem 1.** *If  $M$  and  $N$  are finite graded  $R$ -modules such that  $\text{Ext}_R^i(M, N) = 0$  for  $i \gg 0$ , then there is an equality of rational functions*

$$\sum_i (-1)^i H_{\text{Ext}_R^i(M, N)}(t) = \frac{H_M(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})}.$$

To illustrate the scope of the theorem, we apply it to a well known special case.

*Remark.* For an  $m$ -dimensional graded Cohen-Macaulay module  $M$  over a  $d$ -dimensional graded Cohen-Macaulay ring  $R$  with canonical module  $\omega$ , there is an equality, cf. [5; (4.3.7)],

$$H_{\text{Ext}_R^{d-m}(M, \omega)}(t) = (-1)^m H_M(t^{-1}).$$

Indeed,  $\text{Ext}_R^i(M, \omega) = 0$  unless  $i = d - m$  by [5; (3.3.10)], so we only need to show that  $H_\omega(t) = (-1)^d H_R(t^{-1})$ . Consider the polynomial ring  $Q = K[X_1, \dots, X_e]$  graded by  $\deg X_j = d_j$  and the surjective homomorphism of  $K$ -algebras  $Q \rightarrow R$  with  $X_j \mapsto x_j$  for  $1 \leq j \leq e$ . By [5; (3.6.12), (3.6.10)], for  $W = Q(-\sum_{i=1}^e d_i)$  we have  $\omega \cong \text{Ext}_Q^{e-d}(R, W)$ , hence we get  $H_\omega(t)$  by applying the theorem to the  $Q$ -modules  $R$  and  $W$ .

Next we record some general properties of Laurent expansions.

*Remark.* When  $f(t)$  is a function of a complex variable  $t$ , we denote by  $[f(t)]_z$  the expansion of  $f(t)$  as a Laurent series  $\sum_{j \in \mathbb{Z}} a_j (t - z)^j$  when  $z \in \mathbb{C}$ , and as a Laurent series  $\sum_{j \in \mathbb{Z}} a_j t^{-j}$  when  $z = \infty$ . The *order* of  $[f(t)]_z$  is introduced by  $\text{ord}[f(t)]_z = \inf\{j \in \mathbb{Z} \mid a_j \neq 0\}$ .

For a finite graded  $R$ -module  $M$  we often write  $[M]_z$  in place of  $[H_M(t)]_z$ . Clearly,  $[M]_0 \in \mathbb{Z}((t))$  and  $[M]_\infty \in \mathbb{Z}((t^{-1}))$ , where  $\mathbb{A}((t)) = \mathbb{A}[[t]][[t^{-1}]]$  and  $\mathbb{A}((t^{-1})) = \mathbb{A}[[t^{-1}]][[t]]$  denote rings of formal Laurent series of finite order with coefficients in  $\mathbb{A}$ , with topologies

defined by the powers of the respective indeterminate. Note that if  $\{g_i\}_{i \geq 0}$  is a sequence of formal Laurent series in either ring, then  $\sum_{i \geq 0} g_i$  converges when  $\lim_{i \rightarrow \infty} \text{ord } g_i = \infty$ .

If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence of graded  $R$ -modules, then  $[M]_z = [M']_z + [M'']_z$ . Indeed, additivity is obvious for  $z = 0$ , hence yields an equality of rational functions  $H_M(t) = H_{M'}(t) + H_{M''}(t)$ , which implies additivity for arbitrary  $z$ .

Similarly, one sees that  $[M(a)]_z = [t^{-a}]_z [M]_z$  for any  $a \in \mathbb{Z}$ , where  $M(a)$  is the  $a$ 'th translate of  $M$ , that is, the graded  $R$ -module with  $M(a)_n = M_{a+n}$  for  $n \in \mathbb{Z}$ .

Without homological assumptions on  $M$  or  $N$ , we establish the following limited

**Proposition 2.** *If  $M$  and  $N$  are finite graded  $R$ -modules, and  $N$  has finite length, then  $H_{\text{Ext}_R^i(M, N)}(t) \in \mathbb{Z}[t, t^{-1}]$ , the order of  $[\text{Ext}_R^i(M, N)]_\infty$  goes to infinity with  $i$ , and  $\sum_i (-1)^i [\text{Ext}_R^i(M, N)]_\infty \in \mathbb{Z}((t^{-1}))$  is the Laurent expansion around  $\infty$  of  $\phi_R(M, N)(t)$ .*

For comparison and for later application, we recall the situation in homology. The grading of  $M \otimes_K N$  by  $(M \otimes_K N)_n = \bigoplus_{j \in \mathbb{Z}} (M_j \otimes_K N_{n-j})$  induces a grading of  $M \otimes_R N$ . As above, this defines a grading of  $\text{Tor}_i^R(M, N)$  for each  $i$ , and by [1; Lemma 7] we have

**Lemma 3.** *If  $M$  and  $N$  are finite graded  $R$ -modules, then the order of  $[\text{Tor}_i^R(M, N)]_0 \in \mathbb{Z}((t))$  goes to  $\infty$  with  $i$ , and  $\sum_i (-1)^i [\text{Tor}_i^R(M, N)]_0 \in \mathbb{Z}((t))$  is the Laurent expansion around 0 of the rational function*

$$\chi^R(M, N)(t) = \frac{H_M(t) \cdot H_N(t)}{H_R(t)}. \quad \square$$

*Proof of Theorem 1.* Let  $Q \rightarrow R$  and  $W$  be as in the remark following the statement of the theorem, let  $I^\bullet$  be a minimal (hence finite, cf. [5, (3.6.6)]) graded injective resolution of the  $Q$ -module  $W$ , and let  $F^\bullet$  be a minimal graded free resolution of the  $R$ -module  $M$ . As in [2, (4.4.I)], there is a canonical isomorphism of complexes of graded  $R$ -modules

$$\theta: F^\bullet \otimes_R \text{Hom}_Q(N, I^\bullet) \rightarrow \text{Hom}_Q(\text{Hom}_R(F^\bullet, N), I^\bullet)$$

given by  $\theta(f \otimes_R \alpha)(\beta) = (-1)^{i(j-h)} \alpha \beta(f)$  for  $f \in F^i$ ,  $\alpha \in \text{Hom}_Q(N, I^j)$ ,  $\beta \in \text{Hom}_Q(F^h, N)$ . Filtering the complex on the left by the cohomological degree of  $F^\bullet$ , and the one on the right by that of  $I^\bullet$ , we obtain two spectral sequences which converge to a common limit:

$${}_2E_p^q = \text{Tor}_p^R(M, \text{Ext}_Q^q(N, W)) \implies E_{p-q} \longleftarrow \text{Ext}_Q^q(\text{Ext}_R^p(M, N), W) = {}_2E^{p,q}.$$

In the first spectral sequence we have  ${}_r d_p^q: {}_r E_p^q \rightarrow {}_r E_{p-r}^{q-r+1}$  for  $r \geq 2$ , and  ${}_2 E_p^q = 0$  unless  $p \geq 0$  and  $0 \leq q \leq e$ . It follows that  $E_i = 0$  for  $i < -q$ , and that  ${}_{e+2} E_p^q = {}_\infty E_p^q$ .

By Lemma 3 we know that for any fixed  $q$  the order of  $[\text{Tor}_p^R(M, \text{Ext}_Q^q(N, W))]_0 \in \mathbb{Z}((t))$  goes to infinity together with  $p$ , hence  $\sum_p (-1)^p [{}_2 E_p^q]_0$  is in  $\mathbb{Z}((t))$ . As  ${}_r E_p^q$  is a subquotient of  ${}_2 E_p^q$ , we further have  $\sum_p (-1)^p [{}_r E_p^q]_0 \in \mathbb{Z}((t))$  for each  $r \geq 2$ , and we set

$$\chi({}_r E) = \sum_{q=0}^e (-1)^q \sum_p (-1)^p [{}_r E_p^q]_0.$$

Next we show that  $\chi({}_r\mathbf{E}) = \chi({}_{r+1}\mathbf{E})$  for  $r \geq 2$ . To this end, we first note that up to any given degree  $n$  each series is determined by a finite number of the summands used to define it, hence when looking at the coefficient of  $t^n$  we may restrict both summations to the same finite set of indices  $p$  and  $q$ . In  $\chi({}_r\mathbf{E})$  this coefficient is the Euler characteristic of the degree  $n$  part of a finite complex of graded  $R$ -modules. As the differential  ${}_r d_p^q$  preserves the grading of these modules, the coefficient of  $t^n$  in  $\chi({}_{r+1}\mathbf{E})$  is the Euler characteristic of the degree  $n$  part of the homology of this complex. The classical permanence property of Euler characteristics implies that these coefficients are equal.

By the finite convergence of the spectral sequence, we thus see that the series  $\chi(E) = \sum_i (-1)^i [E_i]_0$  is equal to  $\chi({}_2\mathbf{E})$ . Together with Lemma 3 this implies

$$\begin{aligned} \chi(E) &= \sum_{q=0}^e (-1)^q \sum_p (-1)^p \left[ \text{Tor}_p^R(M, \text{Ext}_Q^q(N, W)) \right]_0 \\ &= \sum_{q=0}^e (-1)^q \left[ \chi_R(M, \text{Ext}_Q^q(N, W))(t) \right]_0 \\ &= \left[ \frac{H_M(t)}{H_R(t)} \sum_{q=0}^e (-1)^q H_{\text{Ext}_Q^q(N, W)}(t) \right]_0 . \end{aligned}$$

In order to finish the computation we use the equality of rational functions

$$(2) \quad \sum_{q=0}^e (-1)^q H_{\text{Ext}_Q^q(A, W)}(t) = (-1)^e H_A(t^{-1}) .$$

which holds for each finite  $Q$ -module  $A$ . Indeed, if  $A = Q(b)$ , then the only non-vanishing Ext is  $\text{Ext}_Q^0(Q(b), W) \cong Q(-b - \sum_{j=1}^e d_j)$ , and the equality is checked by a direct computation which uses the expression  $H_Q(t) = 1 / \prod_{j=1}^e (1 - t^{d_j})$ . The general case follows, as each  $A$  has a finite resolution by finite direct sums of translates of  $Q$ , and both sides of the formula are additive functions on the Grothendieck group  $G(Q)$  of the category of finite graded  $Q$ -modules and homomorphisms of degree 0. Thus, we have established that

$$\chi(E) = (-1)^e \left[ \frac{H_M(t) \cdot H_N(t^{-1})}{H_R(t)} \right]_0 = (-1)^e [\phi_R(M, N)(t^{-1})]_0 .$$

Now we turn to the second spectral sequence. It has  ${}^2\mathbf{E}^{p,q} = 0$  unless  $p \geq 0$  and  $0 \leq q \leq e$ , and differentials  ${}^r d^{p,q}: {}^r\mathbf{E}^{p,q} \rightarrow {}^r\mathbf{E}^{p-r+1, q-r}$ , hence  ${}^{e+2}\mathbf{E}^{p,q} = {}^\infty\mathbf{E}^{p,q}$ . Furthermore, our assumption implies that  ${}^r\mathbf{E}^{p,q} = 0$  for  $p \gg 0$ , hence for each  $r \geq 2$  the sum

$$\chi({}_r\mathbf{E}) = \sum_p (-1)^p \sum_{q=0}^e (-1)^q [{}^r\mathbf{E}^{p,q}]_0$$

is actually *finite*. This time we may apply directly the classical argument on Euler characteristics. In view of formula (2) it yields

$$\begin{aligned}\chi(E) &= \sum_p (-1)^p \sum_{q=0}^e (-1)^q \left[ \text{Ext}_Q^q(\text{Ext}_R^p(M, N), W) \right]_0 \\ &= \sum_p (-1)^p \left[ \sum_{q=0}^e (-1)^q H_{\text{Ext}_Q^q(\text{Ext}_R^p(M, N), W)}(t) \right]_0 \\ &= (-1)^e \left[ \sum_p (-1)^p H_{\text{Ext}_R^p(M, N)}(t^{-1}) \right]_0.\end{aligned}$$

The overall result of the preceding computations now reads

$$\left[ \sum_p (-1)^p H_{\text{Ext}_R^p(M, N)}(t^{-1}) \right]_0 = [\phi_R(M, N)(t^{-1})]_0$$

and this clearly implies the desired equality of rational functions.  $\square$

*Proof of Proposition 2.* Consider  $N^\vee = \text{Hom}_K(N, K)$  with the induced structure of graded  $R$ -module. By Lemma 3, the expansion around 0 of the rational function  $\chi^R(M, N)(t)$  is equal to  $\sum_i (-1)^i [\text{Tor}_i^R(M, N^\vee)]_0 \in \mathbb{Z}((t))$ . As  $\text{rank}_K N$  is finite, we have  $H_{N^\vee}(t) = H_N(t^{-1}) \in \mathbb{Z}[t, t^{-1}]$ , and thus an equality of formal Laurent series

$$\sum_{i \geq 0} (-1)^i \left[ H_{\text{Tor}_i^R(M, N^\vee)}(t) \right]_0 = \left[ \frac{H_M(t) \cdot H_N(t^{-1})}{H_R(t)} \right]_0.$$

In the commutative diagram of homomorphisms of rings

$$\begin{array}{ccc} \mathbb{C}(t) & \xrightarrow{t \mapsto t^{-1}} & \mathbb{C}(t) \\ \downarrow [\ ]_0 & & \downarrow [\ ]_\infty \\ \mathbb{C}((t)) & \xrightarrow{t \mapsto t^{-1}} & \mathbb{C}((t^{-1})) \end{array}$$

the lower row is an isomorphism of topological fields, hence the preceding equality yields

$$\sum_{i \geq 0} (-1)^i \left[ H_{\text{Tor}_i^R(M, N^\vee)}(t^{-1}) \right]_\infty = \left[ \frac{H_M(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})} \right]_\infty.$$

The canonical isomorphisms  $(M \otimes_R (N^\vee))^\vee \cong \text{Hom}_R(M, N^{\vee\vee}) \cong \text{Hom}_R(M, N)$  extend to isomorphisms of graded  $R$ -modules  $(\text{Tor}_i^R(M, N^\vee))^\vee \cong \text{Ext}_R^i(M, N)$  for  $i \geq 0$ . Thus, in  $\mathbb{Z}[t, t^{-1}]$  we have  $H_{\text{Tor}_i^R(M, N^\vee)}(t^{-1}) = H_{\text{Ext}_R^i(M, N)}(t)$ , and this finishes the proof.  $\square$

2. BASS NUMBERS

Let  $N$  be a finite graded  $R$ -module. For an integer  $i$  and a prime ideal  $\mathfrak{p}$  of  $R$ , the rank of  $\text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), N_{\mathfrak{p}})$  over the field  $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$  is known as the  $i$ 'th *Bass number* of  $N$  and is denoted  $\mu_R^i(\mathfrak{p}, N)$ . We establish nontrivial lower bounds for the Bass numbers  $\mu_R^i(N) = \mu_R^i(R_+, N)$  of “many” graded modules of finite injective dimension.

A graded ring  $R$  is said to be *standard* if it is generated by elements of degree 1. In this case  $H_N(t)$  has a unique expression of the form

$$(3) \quad H_N(t) = \frac{e_N(t)}{(1-t)^n},$$

where  $n = \dim N$ , and  $e_N(t)$  is a Laurent polynomial in  $\mathbb{Z}[t, t^{-1}]$  such that  $e_N(1)$  is a positive integer, known as the *multiplicity* of  $N$ .

**Theorem 4.** *Let  $R$  be a standard graded ring of Krull dimension  $d$ .*

*If  $N \neq 0$  is a finite graded  $R$ -module of finite injective dimension and Krull dimension  $n$ , then  $e_R(t^{-1})$  divides  $e_N(t)$  in  $\mathbb{Z}[t, t^{-1}]$ . If  $p$  is a prime number and  $q$  is defined by*

$$q = \max \left\{ p^r \in \mathbb{Z} \mid \sum_{s=0}^{p^r} t^s \text{ divides } \frac{e_N(t)}{e_R(t^{-1})} \text{ in } \mathbb{Z}[t, t^{-1}] \right\},$$

*then the Bass numbers of  $N$  satisfy the inequality*

$$\sum_i \mu_R^i(N) \geq p^{\frac{d-n+q-1}{q(p-1)}}.$$

*Proof.* Since each  $\text{Ext}_R^i(K, N)$  is a finite graded  $K$ -vector space, and is trivial for  $i > \text{injdim}_R N$ , the rational function  $\sum_i (-1)^i H_{\text{Ext}_R^i(K, N)}(t)$  is a Laurent polynomial, which we denote by  $\phi_R^N(t)$ . Theorem 1 and formula (3) yield an equality in  $\mathbb{Z}[t, t^{-1}]$ :

$$(1-t)^{d-n} e_N(t) = (-t)^d e_R(t^{-1}) \phi_R^N(t).$$

As  $d \geq n$  and  $e_R(1) \neq 0$ , unique factorization in  $\mathbb{Z}[t, t^{-1}]$  shows that  $e_R(t^{-1})$  divides  $e_N(t)$ . This proves the first assertion. The second one follows by the argument of [1; §4].  $\square$

Under the hypotheses of the next result the theorem applies with  $p = 2$  and  $q = 1$ :

**Corollary 5.** *If  $-1$  is not a root of the Laurent polynomial  $e_N(t)/e_R(t^{-1})$ , then*

$$\sum_i \mu_R^i(N) \geq 2^{d-n}.$$

*This inequality holds, in particular, when the multiplicity of  $N$  is an odd number.*  $\square$

*Remark.* If  $N$  is a finite module over a local ring  $(S, \mathfrak{n})$ , and  $\text{depth}_S N = g < d = \dim S$ , then a result of Bruns [5; (9.6.1.a)] implies that  $\sum_i \mu_S^i(N) \geq (d-g)(d-g-1) + 3$ .

The *quadratic* lower bound above is in general weaker than the *exponential* bound given by the corollary, so we raise the question whether an inequality  $\sum_i \mu_S^i(\mathfrak{n}, N) \geq 2^{d-n}$  holds for each finite  $S$ -module  $N$  of Krull dimension  $n$  and of *finite injective dimension*. Such a bound would be best possible: if  $\mathfrak{x}$  is a system of parameters of  $S$ ,  $E$  is the injective envelope of  $S/\mathfrak{n}$ , and  $N$  is the finite length module  $\text{Hom}_S(S/(\mathfrak{x}), E)$ , then  $\mu_S^i(\mathfrak{n}, N) = \binom{d}{i}$  since  $S$  is Cohen–Macaulay by the Bass Conjecture proved by Peskine, Szpiro, Hochster, and P. Roberts, cf. [5; (9.6.2), (9.6.4.b)].

### 3. LAURENT COEFFICIENTS

We turn to properties of the coefficients of Laurent expansions around 1 of rational functions representing the Hilbert series of finite  $R$ -modules. For purposes of calibration, it is convenient to write such an expansion in the form

$$[M]_1 = \sum_{j \geq 0} \frac{f_R^j(M)}{(1-t)^{d-j}}$$

with  $d$  equal to the Krull dimension of  $R$ , and to set  $f_R^j(M) = 0$  for  $j < 0$ . We call  $f_R^j(M)$  the  $j$ 'th *Laurent coefficient* of  $M$  and note that it is a rational number.

*Remark.* When  $R$  is standard and  $d - \dim M = h$ , comparison with (3) yields  $\text{rank}_K(M_n) = \sum_{j=h}^{d-1} f_R^j(M) \binom{n+d-j-1}{d-j-1}$  for  $n \gg 0$ , hence the Laurent coefficients of the principal part of  $[M]_1$  are directly related to the coefficients of the Hilbert polynomial of  $M$  in a familiar “binomial” format, e.g. [5; (4.1.5)]:  $f_R^j(M) = (-1)^{h-j} e_{h-j}(M)$  for  $h \leq j \leq d-1$ .

Under the assumptions of Theorem 1, the  $j$ 'th coefficient in the Laurent expansion around 1 of the rational function  $\phi_R(M, N)(t)$  is an alternating sum of the corresponding Laurent coefficients of Ext modules. The number

$$\varepsilon_R^j(M, N) = \sum_{i=0}^j (-1)^i f_R^j(\text{Ext}_R^i(M, N))$$

represents a portion of such a sum. The next result shows that under a homological finiteness hypothesis it contains the entire information.

**Proposition 6.** *If the projective dimension of a finite graded  $R$ -module  $M$  or the injective dimension of a finite graded  $R$ -module  $N$  is finite, then*

$$[\phi_R(M, N)(t)]_1 = \sum_{j \geq 0} \frac{\varepsilon_R^j(M, N)}{(1-t)^{d-j}}.$$

The proposition applies to all finite modules over a graded polynomial ring  $R$ , and raises the question of extending its validity—at least in part—to other graded rings. When

$$\phi_R(M, N)(t) = \sum_{j=0}^n \frac{\varepsilon_R^j(M, N)}{(1-t)^{d-j}} + O\left(\frac{1}{(1-t)^{d-n-1}}\right)$$

for some integer  $n \geq 0$ , we say that the sequences  $\phi_R^*(M, N)$  (of coefficients of the Laurent expansion of  $\phi_R(M, N)(t)$  around 1) and  $\varepsilon_R^*(M, N) = \{\varepsilon_R^j(M, N)\}_{j \in \mathbb{Z}}$  agree up to level  $n$ . Rewriting the condition in the form

$$H_R(t^{-1}) \cdot \sum_{j=0}^n \frac{\varepsilon_R^j(M, N)}{(1-t)^{d-j}} = H_M(t^{-1}) \cdot H_N(t) + O\left(\frac{1}{(1-t)^{2d-n-1}}\right)$$

and taking Laurent expansions around 1 on both sides, we see it is equivalent to a system of  $n+1$  numerical equalities involving the numbers  $f_R^j(M)$ ,  $f_R^j(N)$ ,  $f_R^j(R)$ , and  $\varepsilon_R^j(M, N)$  for  $0 \leq j \leq n$ . For  $n=2$  such a system is displayed below in equations (4.0) through (4.2).

We adopt the convention that the *codimension* of a prime ideal  $\mathfrak{p}$  of  $R$  is the non-negative integer  $\text{codim } \mathfrak{p} = \dim R - \dim(R/\mathfrak{p})$  and note that  $\text{height } \mathfrak{p} \leq \text{codim } \mathfrak{p}$ , with equality when  $R$  is an integral domain. The ring  $R$  is *regular in codimension  $c$*  if  $R_{\mathfrak{p}}$  is regular for each prime  $\mathfrak{p}$  such that  $\text{codim } \mathfrak{p} \leq c$ ; by Matijevic [9; (2.1)] it suffices to impose the condition only on the primes in  $\text{Proj } R$ , the set of *homogeneous* prime ideals of  $R$ .

**Theorem 7.** *Let  $M$  and  $N$  be finite graded modules over a graded ring  $R$ .*

*If  $R$  has a unique prime  $\mathfrak{p}$  of codimension 0 and  $R_{\mathfrak{p}}$  is a field, then*

$$(4.0) \quad f_R^0(R)\varepsilon_R^0(M, N) = f_R^0(M)f_R^0(N).$$

*If  $R$  is an integral domain which is regular in codimension 1, then*

$$(4.1) \quad f_R^0(R)\varepsilon_R^1(M, N) - f_R^1(R)\varepsilon_R^0(M, N) = f_R^0(M)f_R^1(N) - f_R^1(M)f_R^0(N).$$

*If  $R$  is a unique factorization domain which is regular in codimension 2, then*

$$(4.2) \quad f_R^0(R)\varepsilon_R^2(M, N) - f_R^1(R)\varepsilon_R^1(M, N) + (f_R^2(R) - f_R^1(R))\varepsilon_R^0(M, N) = f_R^0(M)f_R^2(N) - f_R^1(M)f_R^1(N) + (f_R^2(M) - f_R^1(M))f_R^0(N).$$

*Remark.* When  $R$  is a domain for any finite graded  $R$ -module  $M$  one has

$$(5) \quad f_R^0(M) = \text{rank}_R(M)f_R^0(R),$$

cf. Lemma 9(c) below. If furthermore  $R$  is regular in codimension 1, then formula (1), due to Benson and Crawley-Boevey [4; (2.4)] or [3; (3.3.2)], follows from (4.0) and (4.1).

Recall that a *graded complete intersection* is a quotient of a graded polynomial ring by a regular sequence of homogeneous elements. By a classical result of Grothendieck [8; (XI.3.14)], cf. also [6], a complete intersection which is regular in codimension 3 is factorial, hence we get the following easily applicable

**Corollary 8.** *If  $R$  is a graded complete intersection which is regular in codimension 3, then the equalities of the theorem hold for all finite graded  $R$ -modules.  $\square$*

In the next section we show that neither the UFD hypothesis of the theorem nor the codimension 3 hypothesis of the corollary can be significantly weakened. The proofs of the results above depend on a few lemmas; the first one (for  $\text{deg} = f_R^0$  and  $\psi = f_R^1$ ) appears in [4; (2.2), (2.3)], cf. also [3; (2.4.1)].

**Lemma 9.** *Let  $M \neq 0$  be a finite  $R$ -modules with  $h = \dim R - \dim M$ .*

- (a)  $f_R^j(-)$  is an additive function on  $G(R)$  for each  $j \in \mathbb{Z}$ .
- (b)  $f_R^j(M) = 0$  for  $j < h$  and  $f_R^h(M) > 0$ .
- (c)  $f_R^h(M) = \sum_{\substack{\mathfrak{p} \in \text{Proj } R \\ \text{codim } \mathfrak{p} = h}} \text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) f_R^h(R/\mathfrak{p})$ .

*Proof.* (a) results from the additivity of  $[\ ]_1$  and the uniqueness of Laurent expansions.

(b) is a fact of dimension theory.

(c) is a formal consequence of the preceding two because  $M$  admits a finite filtration with subquotients of the form  $(R/\mathfrak{q})(a)$  for appropriate  $\mathfrak{q} \in \text{Proj } R$  and  $a \in \mathbb{Z}$ .  $\square$

**Lemma 10.** *Let  $M$  and  $N$  be finite  $R$ -modules, such that for some  $c \in \mathbb{N}$  and for all  $\mathfrak{p} \in \text{Proj } R$  with  $\text{codim } \mathfrak{p} \leq c$  either  $\text{proj dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$  or  $\text{inj dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$  is finite. For each integer  $j \leq c$  and any submodule  $L$  of  $\text{Ext}_R^i(M, N)$ , we have  $f_R^j(L) = 0$  when  $i > j$ .*

*Proof.* By Lemma 9(b) it suffices to show that  $j < \dim R - \dim L = g$ , hence we may assume that  $g \leq c$ . If  $\mathfrak{p} \in \text{Supp } L$  is a homogeneous prime with  $\text{codim } \mathfrak{p} = g$ , then

$$\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong \text{Ext}_R^i(M, N)_{\mathfrak{p}} \supseteq L_{\mathfrak{p}} \neq 0$$

implies that  $i \leq \min\{\text{proj dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}), \text{inj dim}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})\}$ . One of these dimensions is finite by assumption, and thus is bounded above by  $\text{depth } R_{\mathfrak{p}}$ , cf. [5; (1.3.3) and (3.1.17)]. Thus, we have  $j < i \leq \text{depth } R_{\mathfrak{p}} \leq \text{codim } \mathfrak{p} = g$ , as desired.  $\square$

*Proof of Proposition 6:* Equate the Laurent expansions around 1 of the rational functions in Theorem 1 and use Lemma 10 to regroup the terms in the sum of Ext's.  $\square$

The argument for part (a) of the next lemma is abstracted from the proof of [4; (2.4)].

**Lemma 11.** *The numbers  $\varepsilon_R^j(M, N)$  have the following properties.*

- (a)  $\varepsilon_R^j(-, -)$  is a biadditive function on  $G(R)$  if  $R$  is regular in codimension  $c$  and  $j \leq c$ .
- (b)  $\varepsilon_R^j(M, N) = 0$  when  $j < \dim R - \dim(R/(\text{ann } M + \text{ann } N))$ .

*Proof.* (a) Fix an integer  $j$  between 0 and  $c$ , take a short exact sequence of finite  $R$ -modules  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ , and consider the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(M, N') \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N'') \rightarrow \\ \text{Ext}_R^1(M, N') \rightarrow \dots \rightarrow \text{Ext}_R^{j-1}(M, N'') \rightarrow \\ \text{Ext}_R^j(M, N') \rightarrow \text{Ext}_R^j(M, N) \rightarrow \text{Ext}_R^j(M, N'') \rightarrow N^{j+1} \rightarrow 0. \end{aligned}$$

By assumption, the ring  $R_{\mathfrak{p}}$  has global dimension  $\leq c$  whenever  $\text{codim } \mathfrak{p} \leq c$ , hence Lemma 10 shows that  $f_R^j(N^{j+1}) = 0$ . The additivity of  $\varepsilon_R^j(M, -)$  now follows by applying  $f_R^j$  to the long exact sequence. The additivity of  $\varepsilon_R^j(-, N)$  yields to a similar approach.

(b) For each  $i$  the  $R$ -module  $\text{Ext}_R^i(M, N)$  is annihilated by  $(\text{ann } M + \text{ann } N)$ , hence we have  $\dim R - \dim \text{Ext}_R^i(M, N) \geq \dim R - \dim R/(\text{ann } M + \text{ann } N) > j$ , and thus  $\varepsilon_R^j(M, N)$  vanishes in the indicated range by Lemma 9(b).  $\square$

An agreement of  $\varepsilon^*$  and  $\phi^*$  imposes strong restrictions on the local structure of  $R$ :

**Lemma 12.** *Let  $\mathfrak{p}$  be a homogeneous prime ideal of  $R$  with  $\text{codim } \mathfrak{p} = h$ . If  $h \geq 1$ , then the sequences  $\varepsilon_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  and  $\phi_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  agree up to level  $h$  precisely when the local ring  $R_{\mathfrak{p}}$  is regular; if  $h = 0$ , then they agree up to level 0 if and only if  $R_{\mathfrak{p}}$  is a field and  $\mathfrak{p}$  is the unique prime of codimension 0 in  $R$ .*

*Proof.* Lemma 9(b) shows that  $f_R^h(R/\mathfrak{p})$  and  $f_R^0(R)$  are positive, and that

$$[\phi_R(R/\mathfrak{p}, R/\mathfrak{p})(t)]_1 = (-1)^h \cdot \frac{f_R^h(R/\mathfrak{p})}{f_R^0(R)} \cdot \frac{f_R^h(R/\mathfrak{p})}{(1-t)^{d-2h}} + O\left(\frac{1}{(1-t)^{d-2h-1}}\right).$$

With  $\eta_h(\mathfrak{p}) = \sum_{i=0}^h (-1)^i \text{rank}_{k(\mathfrak{p})}(\text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), k(\mathfrak{p})))$ , Lemmas 11(b) and 9(c) give

$$\sum_{j=0}^h \frac{\varepsilon_R^j(R/\mathfrak{p}, R/\mathfrak{p})}{(1-t)^{d-j}} = \eta_h(\mathfrak{p}) \cdot \frac{f_R^h(R/\mathfrak{p})}{(1-t)^{d-h}}.$$

Thus,  $\varepsilon_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  and  $\phi_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  agree up to level 0 if and only if  $f_R^0(R) = f_R^0(R/\mathfrak{p})$ . In view of Lemma 9(c), this is equivalent to the statement that  $\mathfrak{p}$  is the only prime of  $R$  with  $\text{codim } \mathfrak{p} = 0$  and that  $R_{\mathfrak{p}}$  is a field.

If  $h > 0$ , then  $\varepsilon_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  and  $\phi_R^*(R/\mathfrak{p}, R/\mathfrak{p})$  agree up to level  $h$  if and only if  $\eta_h(\mathfrak{p}) = 0$ . When the local ring  $R_{\mathfrak{p}}$  is regular the rank of  $\text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), k(\mathfrak{p}))$  over  $k(\mathfrak{p})$  is equal to  $\binom{h}{i}$ , hence  $\eta_h(\mathfrak{p}) = 0$ . When  $R_{\mathfrak{p}}$  is not regular its embedding dimension is at least 2 because  $\dim R_{\mathfrak{p}} = h \geq 1$ , and then  $\eta_h(\mathfrak{p}) \neq 0$  by Okiyama [10; Proposition 8].  $\square$

**Lemma 13.** (a) *If  $M$  and  $N$  are finite graded  $R$ -modules, then for  $a, b \in \mathbb{Z}$  the sequences  $\varepsilon_R^*(M(a), N(b))$  and  $\phi_R^*(M(a), N(b))$  agree up to level  $n$  if and only if  $\varepsilon_R^*(M, N)$  and  $\phi_R^*(M, N)$  agree up to the same level.*

(b) *If  $\mathfrak{p}$  is a homogeneous prime ideal such that  $\text{codim } \mathfrak{p} = h$  and the local ring  $R_{\mathfrak{p}}$  is Gorenstein, then  $\varepsilon_R^*(R/\mathfrak{p}, R)$  and  $\phi_R^*(R/\mathfrak{p}, R)$  agree up to level  $h$ .*

*Proof.* (a) The canonical isomorphisms  $\text{Ext}_R^i(M(a), N(b)) \cong \text{Ext}_R^i(M, N)(b-a)$  show that

$$\sum_{j=0}^n \frac{\varepsilon_R^j(M(a), N(b))}{(1-t)^{d-j}} = t^{a-b} \cdot \sum_{j=0}^n \frac{\varepsilon_R^j(M, N)}{(1-t)^{d-j}}$$

holds for each  $n \geq 0$ , while it is clear that  $\phi_R(M(a), N(b))(t) = t^{a-b} \cdot \phi_R(M, N)(t)$ .

(b) By a direct computation as in the proof of Lemma 12, we get

$$[\phi_R(R/\mathfrak{p}, R)(t)]_1 = (-1)^h \cdot \frac{f_R^h(R/\mathfrak{p})}{(1-t)^{d-h}} + O\left(\frac{1}{(1-t)^{d-h-1}}\right).$$

On the other hand, the Gorenstein hypothesis means that  $\mu_R^i(\mathfrak{p}, R) = \delta_{ih}$ , thus  $\varepsilon_R^j(R/\mathfrak{p}, R) = 0$  for  $j < h$ , while  $\varepsilon_R^h(R/\mathfrak{p}, R) = (-1)^h f_R^h(R/\mathfrak{p})$  by Lemma 9(c), so that

$$\sum_{j=0}^h \frac{\varepsilon_R^j(R/\mathfrak{p}, R)}{(1-t)^{d-j}} = (-1)^h \cdot \frac{f_R^h(R/\mathfrak{p})}{(1-t)^{d-h}}. \quad \square$$

*Proof of Theorem 7.* Due to Lemma 9(a) and 11(a), both sides of (4.n) are biadditive functions on  $G(R)$  for  $n = 0, 1, 2$ . Thus, it suffices to consider  $M$  and  $N$  which are translates of quotients of  $R$  by homogeneous prime ideals. By Lemma 13(a), we can even assume that  $M = R/\mathfrak{p}$  and  $N = R/\mathfrak{q}$  with  $\mathfrak{p}, \mathfrak{q} \in \text{Proj } R$ .

If  $\text{codim } \mathfrak{p} > n$ , or  $\text{codim } \mathfrak{q} > n$ , or  $\text{codim } \mathfrak{p} = n = \text{codim } \mathfrak{q}$  and  $\mathfrak{p} \neq \mathfrak{q}$ , then the left hand side of (4.n) vanishes by Lemma 11(b), and its right hand side vanishes by Lemma 9(b). If  $\text{codim } \mathfrak{p} = n$  and  $\mathfrak{p} = \mathfrak{q}$  (respectively, if  $\mathfrak{p} = 0$  or if  $\mathfrak{q} = 0$ ), then (4.n) holds by Lemma 12 (respectively, by Proposition 6 or by Lemma 13(b)).

It remains to prove (4.2) when  $\text{codim } \mathfrak{r} = 1$ , where  $\mathfrak{r} = \mathfrak{p}$  or  $\mathfrak{r} = \mathfrak{q}$ . As  $R$  is factorial,  $\mathfrak{r} = (x)$  for a homogeneous  $x \in R$ , hence by additivity and Lemma 13(a) we get the result from the exact sequence  $0 \rightarrow R(-\deg x) \rightarrow R \rightarrow R/\mathfrak{r} \rightarrow 0$  and already settled cases.  $\square$

**Remark 14.** Let  $M$  be a finite graded module over a graded integral domain  $R$ .

If  $R$  is Gorenstein in codimension 1, then

$$(6.1) \quad f_R^1(\text{Hom}_R(M, R)) - f_R^1(\text{Ext}_R^1(M, R)) = 2f_R^1(R) \text{rank}_R(M) - f_R^1(M).$$

If  $R$  is factorial, then in addition

$$(6.2) \quad \begin{aligned} & f_R^2(\text{Hom}_R(M, R)) - f_R^2(\text{Ext}_R^1(M, R)) + f_R^2(\text{Ext}_R^2(M, R)) = \\ & \left(1 + 2\frac{f_R^1(R)}{f_R^0(R)}\right) \left(f_R^1(R) \text{rank}_R(M) - f_R^1(M)\right) + f_R^2(M). \end{aligned}$$

Indeed, the formulas above are obtained by rewriting (4.1) and (4.2) for  $N = R$  with the help of (5). A factorial domain is Cohen-Macaulay in codimension 2 (due to normality), and thus is Gorenstein in this codimension by a result of Murthy, cf. [5; (3.3.19)]. The argument for Lemma 11(a) shows that  $\varepsilon_R^j(-, R)$  is an additive function on  $G(R)$  for  $j = 0, 1, 2$ . At this point we may retrace the preceding proof, ignoring any discussion of primes  $\mathfrak{q} \neq 0$ .

#### 4. NORMAL DOMAINS

Our examples are built from the ring  $R = K[X, Y, U, V]/(XV - YU) = K[x, y, u, v]$  with the standard grading, and the  $R$ -module  $M = R/(u, v) \cong K[X, Y]$ .

**Example 15.** Equality (4.2) may fail over a 3-dimensional graded hypersurface ring  $R$  which is regular in codimension 2 (and hence is a normal domain).

Over  $K[X, Y, U, V]$  the quadratic form  $XV - YU$  has a homogeneous *matrix factorization*

$$\begin{pmatrix} V & -Y \\ -U & X \end{pmatrix} \begin{pmatrix} X & Y \\ U & V \end{pmatrix} = \begin{pmatrix} XV - YU & 0 \\ 0 & XV - YU \end{pmatrix} = \begin{pmatrix} X & Y \\ U & V \end{pmatrix} \begin{pmatrix} V & -Y \\ -U & X \end{pmatrix},$$

hence Eisenbud [7; (4.1)] provides the following minimal graded free resolution of  $M$ :

$$\dots \rightarrow R^2(-4) \xrightarrow{\begin{pmatrix} v & -y \\ -u & x \end{pmatrix}} R^2(-3) \xrightarrow{\begin{pmatrix} x & y \\ u & v \end{pmatrix}} R^2(-2) \xrightarrow{\begin{pmatrix} v & -y \\ -u & x \end{pmatrix}} R^2(-1) \xrightarrow{(u, v)} R \rightarrow 0.$$

Applying to it the functor  $\text{Hom}_R(-, M)$  we get a complex of graded  $R$ -modules

$$0 \rightarrow M \xrightarrow{0} M^2(1) \xrightarrow{\begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix}} M^2(2) \xrightarrow{\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}} M^2(3) \xrightarrow{\begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix}} M^2(4) \rightarrow \dots$$

Direct computations based on the  $M$ -regularity of the sequence  $x, y$  yield

$$\begin{aligned} M &\cong \text{Hom}_R(M, M) \cong \text{Ext}_R^1(M, M); \\ \text{Ext}_R^{2n}(M, M) &\cong k(2n) \quad \text{for } n \geq 1; \\ \text{Ext}_R^{2n+1}(M, M) &= 0 \quad \text{for } n \geq 1. \end{aligned}$$

Accordingly, we get Laurent expansions

$$\begin{aligned} [R]_1 &= \left[ \frac{1-t^2}{(1-t)^4} \right]_1 = \frac{2}{(1-t)^3} - \frac{1}{(1-t)^2}; \\ [M]_1 &= [\text{Hom}_R(M, M)]_1 = [\text{Ext}_R^1(M, M)]_1 = \frac{1}{(1-t)^2}; \\ [\text{Ext}_R^2(M, M)]_1 &= \left[ \frac{1}{t^2} \right]_1 = \sum_{j \geq 0} (j+1)(1-t)^j. \end{aligned}$$

It follows that for  $M = N$  the expression on the left hand side of (4.2) is equal to 0, while the one on its right hand side is equal to  $-1$ .

Another interesting feature of this example is revealed by the computation

$$\sum_{i \geq 0} (-1)^i [\text{Ext}_R^i(M, M)]_\infty = \sum_{n \geq 1} t^{-2n} = \left[ \frac{t^{-2}}{(1-t^{-2})} \right]_\infty = [\phi_R(M, M)(t)]_\infty.$$

It shows that the assumption of finite length in Proposition 2 is not always necessary.

**Example 16.** *Equality (6.2) may fail over a 4-dimensional graded Cohen-Macaulay ring  $R'$  of type 2 which is regular in codimension 3 (and hence is a normal domain).*

Let  $R' = R[Z, W]/(xW - uZ, yW - vZ) = K[x, y, z, u, v, w]$  with the standard grading, and set  $M' = R'/(u, v, w) \cong K[X, Y, Z]$ . Since  $w$  is  $R'$ -regular and annihilates  $M'$ , for  $S = R'/(w) \cong R[Z]/(uZ, vZ)$  and for each  $i$  we have  $\text{Ext}_{R'}^i(M', R') \cong \text{Ext}_S^{i-1}(M', S)(1)$ . As  $M' = M[Z]$ , and the minimal free resolution of  $M[Z]$  over  $R[Z]$  is “the same as” that of  $M$  over  $R$ , it is easy to get a beginning of a minimal  $S$ -free resolution of  $M'$  in the form

$$S^4(-2) \xrightarrow{\begin{pmatrix} v & -y & z & 0 \\ -u & x & 0 & z \end{pmatrix}} S^2(-1) \xrightarrow{(u, v)} S \rightarrow 0.$$

Thus, we are interested in the homology of the complex

$$0 \rightarrow S \xrightarrow{\alpha = \begin{pmatrix} u \\ v \end{pmatrix}} S^2(1) \xrightarrow{\beta = \begin{pmatrix} v & -u \\ -y & x \\ z & 0 \\ 0 & z \end{pmatrix}} S^4(2).$$

In degree 0 we have  $\text{Ker } \alpha = (0 :_S (u, v)) = zS \cong M'(-1)$ , hence  $\text{Ext}_{R'}^1(M', R') \cong M'$ .

In degree 1 the complex is exact, hence  $\text{Ext}_{R'}^2(M', R') = 0$ . Indeed, writing  $b \in S^2(1)$  in the form  $(r_1 + s_1z, r_2 + s_2z)$  with uniquely determined  $r_j \in R$ , we see that  $\beta(b) = 0$  implies

$$vr_1 - ur_2 = 0 \quad -yr_1 + xr_2 = 0 \quad z^2s_1 = 0 \quad z^2s_2 = 0.$$

From the last two equations we get

$$s_j \in (0 :_S z^2) = ((0 :_S z) :_S z) = ((u, v) :_S z) = (u, v)S,$$

hence  $b = (r_1, r_2)$ . In view of the preceding example, the first two equations yield an  $a \in R$  such that  $r_1 = au$  and  $r_2 = av$ , hence  $b = a(u, v) \in \text{Im } \alpha$ .

We can now collect the relevant data on Laurent expansions:

$$\begin{aligned} [R']_1 &= \left[ \frac{1 - 3t^2 + 2t^3}{(1-t)^6} \right]_1 = \frac{3}{(1-t)^4} - \frac{2}{(1-t)^3}; \\ [M']_1 &= [\text{Ext}_{R'}^1(M', R')]_1 = \frac{1}{(1-t)^3}; \\ [\text{Hom}_{R'}(M', R')]_1 &= [\text{Ext}_{R'}^2(M', R')]_1 = 0. \end{aligned}$$

They show that the left hand side of (6.2) is equal to 0, and its right hand side to  $\frac{1}{3}$ .

## REFERENCES

1. Avramov, L. L. and Buchweitz, R.-O., *Lower bounds for Betti numbers*, *Compositio Math.* **86** (1993), 147–158.
2. Avramov, L. L. and Foxby, H.-B., *Homological dimensions of unbounded complexes*, *J. Pure. Appl. Algebra* **71** (1991), 129–155.
3. Benson, D. J., *Polynomial invariants of finite groups*, London Math. Soc. Lecture Note Ser., vol. 190, Univ. Press, Cambridge, 1993.
4. Benson, D. J. and Crawley-Boevey, W. W., *A ramification formula for Poincaré series, and a hyperplane formula for modular invariants*, *Bull. London Math. Soc.* **27** (1995), 435–440.
5. Bruns, W. and Herzog, J., *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math., vol. 39, Univ. Press, Cambridge, 1993.
6. Call, F. and Lyubeznik, G., *A simple proof of Grothendieck’s theorem on the parafactoriality of local rings*, *Birational algebra, syzygies, and multiplicities* (W. J. Heinzer, C. L. Huneke, J. D. Sally, eds.), *Contemp. Math.*, vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 15–18.
7. Eisenbud, D., *Homological algebra on a complete intersection, with an application to group representations*, *Trans. Amer. Math. Soc.* **260** (1980), 35–64.
8. Grothendieck, A., *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (SGA 2), *Adv. Stud. Pure Math.*, vol. 2, North-Holland, Amsterdam, 1968.
9. Matijevic, J., *Three local conditions on a graded ring*, *Trans. Amer. Math. Soc.* **205** (1975), 275–284.
10. Okiyama, S., *A local ring is CM if and only if its residue field has a CM syzygy*, *Tokyo J. Math.* **14** (1991), 489–500.

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