

**JACOBIAN CRITERIA FOR COMPLETE INTERSECTIONS.
THE GRADED CASE**

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Dedicated to Professor Ernst Kunz on his sixtieth birthday

ABSTRACT. Let P be a positively graded polynomial ring over a field k of characteristic zero, let I be a homogeneous ideal of P , and set $R = P/I$. The paper investigates the homological properties of some R -modules canonically associated with R , among them the module $\Omega_{R|k}$ of Kähler differentials and the conormal module I/I^2 .

It is shown that a subexponential bound on the Betti numbers of any of these modules implies that I is generated by a P -regular sequence. In particular, the finiteness of the projective dimension of the conormal module implies R is a complete intersection. Similarly, the finiteness of the projective dimension of the differential module implies R is a reduced complete intersection. This provides strong converses to some well-known properties of complete intersections, and establishes special cases of conjectures of Vasconcelos.

The proofs of these results make extensive use of differential graded homological algebra. The crucial step is to show that the morphism of complexes from the minimal cotangent complex $L_{R|k}$ of André and Quillen into the minimal free resolution of the irrelevant maximal ideal \mathfrak{m} of R , which extends the Euler map $\Omega_{R|k} \rightarrow \mathfrak{m}$, is a split embedding of graded R -modules.

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INTRODUCTION

The classical jacobian criterion characterizes the regularity of an affine algebra over a field of characteristic zero by the projectivity of its module of (Kähler) differentials. We investigate the relations between the singularity of a graded algebra and the homological properties of its module of differentials, and of various other “cotangent modules” canonically associated with the algebra.

In this paper k denotes a field of characteristic zero, and a *graded k -algebra* R is a graded commutative algebra generated over k by finitely many elements of positive degree. Such an algebra is a quotient of a positively graded polynomial ring P over k by a homogeneous ideal I . The ring R is said to be a *complete intersection* if I can be generated by a regular sequence. This property does not depend on the choice of presentation. The minimal length of a regular sequence required for such a presentation is called the *codimension* of R and is denoted $\text{codim } R$. It is known [8] that for a reduced complete intersection R (even in the affine case) the projective dimension of the module of differentials $\Omega_{R|k}$ is at most one. We establish a strong converse.

Theorem 1. *For a graded k -algebra R the following conditions are equivalent:*

- (i) R is a reduced complete intersection;
- (ii) $\text{projdim}_R \Omega_{R|k} \leq 1$;
- (iii) $\text{projdim}_R \Omega_{R|k} < \infty$.

A similar characterization of normal complete intersections is given in (2.9); it involves the torsion of the module of differentials. Recall that the *torsion submodule* $t(M)$ of an R -module M consists of all the elements of M annihilated by non-zero-divisors of R .

In order to describe complete intersections which need not be reduced, we recall that the *i th Betti number* of a finite graded R -module M is the dimension $b_i(M)$ of the k -vector space $\text{Tor}_i^R(M, k)$, which is equal to the rank of the i th module in the minimal homogeneous R -free resolution $F(M)$ of M . The rate of growth of the Betti sequence provides a measure of how complex the homological nature of M is. When comparing rates of growth of two sequences $\{b_i\}_{i \geq 0}$ and $\{c_i\}_{i \geq 0}$ we use the notation $b_i \sim c_i$ to indicate that $b_i = c_i d_i$ for some sequence $\{d_i\}_{i \geq 0}$ with $\lim_{i \rightarrow \infty} d_i = 1$. A numerical measure of the asymptotic rate of growth of the Betti sequence of M is provided by the radius of convergence $\rho(M)$ of the *Poincaré series* $P_M(t) = \sum_{i \geq 0} b_i(M) t^i$.

Theorem 2. *For a graded k -algebra R the following conditions are equivalent:*

- (i) R is a complete intersection;
- (ii) $b_i(\Omega_{R|k}) \sim b i^{d-1}$ for some $b \in \mathbb{Q}$ and $d \in \mathbb{N}$ with $b > 0$ and $\text{codim } R \geq d \geq 0$;
- (iii) $b_i(\Omega_{R|k}) \leq q(i)$ for some polynomial $q \in \mathbb{R}[t]$ and all $i \gg 0$;
- (iv) $\rho(\Omega_{R|k}) \geq 1$, that is, the Poincaré series of $\Omega_{R|k}$ converges in the open unit disk.

Furthermore, these conditions are also equivalent to the ones obtained from (ii), (iii), and (iv) by replacing $\Omega_{R|k}$ with $\Omega_{R|k}/t(\Omega_{R|k})$.

If e is the number of variables of the polynomial ring P , then an exact sequence

$$I/I^2 \rightarrow R^e \rightarrow \Omega_{R|k} \rightarrow 0,$$

relates $\Omega_{R|k}$ and the *conormal module* I/I^2 of the embedding $\text{Spec } R \subseteq \text{Spec } P$. Up to a free direct summand, this graded R -module is independent of the choice of the embedding; in particular, its positive Betti numbers are invariants of R . They vanish when R is a complete intersection, as the conormal module is then well known to be free. By Ferrand [8] and Vasconcelos [20] this condition characterizes quite general complete intersections. In the graded case we prove a stronger result, related to a conjecture of Vasconcelos [21].

Theorem 3. *For a graded k -algebra R the following conditions are equivalent:*

- (i) R is a complete intersection;
- (ii) I/I^2 is a free R -module;
- (iii) $\text{projdim}_R I/I^2 < \infty$;
- (iv) $\rho(I/I^2) \geq 1$, that is, the Poincaré series of I/I^2 converges in the open unit disk.

Furthermore, these conditions are also equivalent to the ones obtained from (ii), (iii), and (iv) by replacing I/I^2 with $(I/I^2)/t(I/I^2)$.

In each of the preceding theorems, the crucial implication is obtained by exhibiting a “large” free subcomplex in the minimal free resolution of the module under consideration. Such subcomplexes are obtained from a *minimal cotangent complex* $L_{R|k}$, defined uniquely up to isomorphism by the theory of André [1] and Quillen [17]. The general machine is not really needed, since in Section 1 we describe a particular construction of $L_{R|k}$. We call $C_i = (L_{R|k})_i/(\partial L_{R|k})_i$ the *i th cotangent module* of the k -algebra R , and note that $C_0 \cong \Omega_{R|k}$ and $C_1 \cong I/I^2$.

Let L be a *minimal* complex of graded free R -modules. In this context minimality means that $\partial(L) \subseteq \mathfrak{m}L$, where \mathfrak{m} is the irrelevant maximal ideal of R . An augmentation of L to a graded R -module M extends uniquely up to homotopy to a morphism of complexes α from L to the minimal resolution $F(M)$, and the induced map

$$\alpha \otimes_R k: L \otimes_R k \rightarrow F(M) \otimes_R k$$

is uniquely defined. An augmentation is said to be *essential* if $\alpha \otimes_R k$ is injective. This means that for each $j \geq 0$ the map α_j is an isomorphism of L_j with a direct summand of the graded free R -module $F_j(M)$, hence all higher order relations of M which appear in the complex L survive in the minimal resolution of M .

Theorem 4. *If R is a graded k -algebra, and $L_{[i]}$ is the truncation at level $i \geq 0$ of the minimal cotangent complex $L_{R|k}$, then the canonical augmentations of $L_{[i]}$ to the R -modules $H_0(L_{[i]}) = C_i$ and $C_i/t(C_i)$ are essential.*

The theorem uncovers a portion of the minimal resolution of C_i , and then results of [9], [7], [4] show that this piece is sufficiently large to imply the desired conclusions. However, it is unlikely to provide the entire resolution, except in special cases: indeed, it is conjectured in [11] that if for some i the complex $L_{[i]}$ is acyclic, then R is a complete intersection. The conjecture is known to hold [19] when R is in the linkage class of a complete intersection, or [11] when the Poincaré series of each finite graded R -module is rational, which is the case [5], [13] over rings of small embedding codimension.

The discussion above could well have been carried out in a more general framework, say that of local k -algebras essentially of finite type. In (a very short) Section 3 we examine possible extensions of the preceding results, in the light of conjectures of Vasconcelos [21].

In our proofs, the grading of the k -algebra R appears through an analysis of the homological algebra of the *Euler derivation*, that is, the k -linear endomorphism of R which multiplies each homogeneous element by its degree. The image of this derivation is contained in \mathfrak{m} , and hence it defines a degree zero homogeneous homomorphism of graded R -modules $\omega: \Omega_{R|k} \rightarrow \mathfrak{m}$, called the *Euler homomorphism*.

The key to all theorems described so far is provided by our

Main Theorem. *The composition of the canonical augmentation $L_{R|k} \rightarrow \Omega_{R|k}$ with the Euler homomorphism $\omega: \Omega_{R|k} \rightarrow \mathfrak{m}$ is an essential augmentation.*

Section 1 is entirely devoted to the proof of the last result. The other ones are deduced from it in Section 2.

1. EULER DERIVATIONS

The main idea behind the proof of the Main Theorem is of interest in its own right: The problem is first modeled in a DG (= differential graded) category, and the data obtained there are then descended to the original setup.

We start by introducing a language for describing the process.

(1.1) *Internal gradings.* All rings and modules are *graded* so that they decompose as direct sums of vector subspaces over the (fixed once and forever) field k . Such gradings, henceforth called *internal*, are not reflected in our notation; the term *homogeneous* refers to internal gradings. We call internally graded k -vector spaces *k -spaces*; \otimes stands for tensor product over k , with the usual grading; homomorphisms of k -spaces are homogeneous (of arbitrary internal degree), and the elements of Hom_k are homogeneous homomorphisms.

(1.2) *External gradings.* In a complex of k -spaces F the differential ∂ is homogeneous of degree 0. Each element $a \in F_i$ is assigned *external degree* i ; this is denoted by $|a| = i$. The *i th truncation* of F is the complex $F_{[i]}$ with $(F_{[i]})_j = 0$ for $j < 0$, $(F_{[i]})_j = F_{i+j}$ for $j \geq 0$, and differential $(\partial_{[i]})_j = (-1)^i \partial_{i+j}$.

(1.3) A *DG algebra* A consists of a complex of k -spaces (A, ∂) together with a chain map $A \otimes A \rightarrow A$, called the *product* of A , which is homogeneous of degree 0 and satisfies the usual associativity and unit conditions. Denoting it by juxtaposition of factors, we have $\partial(aa') = \partial(a)a' + (-1)^{|a|} a \partial(a')$ for $a, a' \in A$. We only consider DG algebras A with $A_i = 0$ for $i < 0$, and with (skew-)commutative product: $aa' = (-1)^{|a||a'|} a'a$ for all $a, a' \in A$.

A homomorphism $\varphi: A \rightarrow B$ of DG algebras is a chain map of the underlying complexes of k -spaces, which preserves internal gradings and commutes with the products. The tensor product $A \otimes B$ is a DG algebra with multiplication $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} (aa' \otimes bb')$.

(1.4) A *DG module* F over the DG algebra A is a complex of k -spaces (F, ∂) equipped with a chain map $A \otimes F \rightarrow F$, $a \otimes f \mapsto af$, which is homogeneous of degree 0 and fulfills the usual rules. The tensor product $F \otimes_A G$ of DG A -modules F and G is the

quotient of the complex $F \otimes G$ by the k -subcomplex spanned by all elements of the form $af \otimes g - (-1)^{|a||f|} f \otimes ag$; it has a natural DG A -module structure: $a(f \otimes_A g) = af \otimes_A g$.

A *degree i homomorphism* $\gamma: F \rightarrow G$ of DG A -modules is a family $\gamma = \{\gamma_j \in \text{Hom}_k(F_j, G_{j+i})\}_{j \in \mathbb{Z}}$ such that $\gamma(af) = (-1)^{i|a|} a\gamma(f)$ for $a \in A$ and $f \in F$; these homomorphisms form a k -space $\text{Hom}_A(F, G)_i$. A *morphism* in the category of DG A -modules is a degree zero homomorphism of DG modules which is also chain map; this category is abelian, with kernels, cokernels, and direct sums defined componentwise. A *quasi-isomorphism* is a morphism $\gamma: F \rightarrow G$ for which the map $H(\gamma): H(F) \rightarrow H(G)$ induced in homology is bijective.

A DG algebra (or module) with zero differential is called an *externally graded algebra* (or module). The functor which trivializes differentials is denoted $(-)^{\natural}$. An externally graded module F over an externally graded algebra A is *free on a basis* $\{e_\lambda \in F\}_{\lambda \in \Lambda}$ if each $f \in F$ can be written uniquely as $f = \sum_{\lambda \in \Lambda} a_\lambda e_\lambda$ with $a_\lambda \in A_{|f| - |e_\lambda|}$ and almost all a_λ equal to zero.

We repeatedly use the following simple fact from DG homological algebra.

(1.5) **Lemma.** *Let F be a DG module over a DG algebra A , such that the A^{\natural} -module F^{\natural} is free and $F_i = 0$ for $i \ll 0$. If J is a DG ideal of A and $H(J) = 0$, then the canonical projection $F \rightarrow F/JF$ is a quasi-isomorphism.*

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a basis of F^{\natural} . If Λ contains only one element, then F is isomorphic to A up to a shift of external degrees and a change of sign for the differential, and the assertion is obvious. If Λ is finite, then take e_μ of minimal degree, set $E = Ae_\mu$, $G = F/E$, and consider the exact sequence of DG modules

$$0 \rightarrow E/JE \rightarrow F/JF \rightarrow G/JG \rightarrow 0.$$

As G^{\natural} is A^{\natural} -free on a basis with fewer elements than that of F^{\natural} , the assertion is obtained from the homology exact sequence by induction. Finally, the general case results from the finite one by a standard direct limit argument. \square

(1.6) *Free extensions.* Given a homogeneous cycle z in a DG algebra A , one can embed A into a DG algebra by freely adjoining a variable x such that $\partial x = z$. In this new DG algebra $A[x]$ the cycle z has been killed, that is, has become a boundary.

This process of killing cycles was introduced by Tate [18]. One first constructs the free externally graded k -algebra $k[x]$ generated by a variable x of degree $s = |z| + 1$ as follows: When s is odd, $k[x]_i = 0$ except for $k[x]_0 = k1$ and $k[x]_s = kx$, and the multiplication is defined by $x^2 = 0$. When s is even, $k[x]_i = 0$ unless $i = js$ for some $j \geq 0$, and $k[x]_{js}$ is the k -vector space on the basis element x^j , with $x^0 = 1$ and $x^1 = x$; the multiplication table is $x^j x^h = x^{j+h}$. Next, one specifies an internal grading on $k[x]$ by postulating for x the same internal degree as that of z . Finally, one defines $A[x]$ by $A[x]^{\natural} = A^{\natural} \otimes_k k[x]$, $\partial(a \otimes 1) = \partial(a) \otimes 1$, and $\partial(1 \otimes x) = z \otimes 1$.

A DG algebra obtained by repeated, (possibly transfinite), adjunction of the variables of a set X is called a *free extension* of the DG algebra A , and is denoted $A[X]$. If $\varphi: A \rightarrow B$ is a homomorphism of DG algebras, and $A \hookrightarrow A[X]$ is a free extension, then the condition

$\varphi[X](x) = x$ for $x \in X$ extends φ to a homomorphism $\varphi[X]: A[X] \rightarrow B[X]$ of DG algebras, where the free extension $B \hookrightarrow B[X]$ is defined by $\partial\varphi[X](x) = \varphi(\partial x)$ for $x \in X$.

Both for illustrative purposes and for later reference we now record:

(1.7) **Lemma.** (a) *If φ is a quasi-isomorphism, then so is $\varphi[X]$.*

(b) *Consider k as DG algebra with trivial grading and differential, let $Y = \{y_\lambda\}_{\lambda \in \Lambda}$ and $\tilde{Y} = \{\tilde{y}_\lambda\}_{\lambda \in \Lambda}$ be disjoint sets of variables such that $|\tilde{y}_\lambda| = |y_\lambda| + 1$ for $\lambda \in \Lambda$, and let $\eta: k \hookrightarrow k[Y, \tilde{Y}]$ be the free extension defined by $\partial(y_\lambda) = 0$ and $\partial(\tilde{y}_\lambda) = y_\lambda$ for $\lambda \in \Lambda$. The DG ideal $(Y, \tilde{Y})k[Y, \tilde{Y}]$ then has trivial homology.*

Proof. (a) The monomials in X of external degree at most p generate DG ideals $\mathcal{F}_p(A)$ and $\mathcal{F}_p(B)$ in $A[X]$ and $B[X]$, respectively. As $\varphi[X](\mathcal{F}_p(A)) \subseteq \mathcal{F}_p(B)$, the homomorphism $\varphi[X]$ induces a homomorphism $\{{}^r\mathbf{E}(\varphi): {}^r\mathbf{E}(A) \rightarrow {}^r\mathbf{E}(B)\}_{r \geq 0}$ of the spectral sequences associated with these filtrations. Note that ${}^0\mathbf{E}(A)$ is the DG algebra with ${}^0\mathbf{E}(A)^\natural = A^\natural[X]$ and differential which extends that of A and is trivial on X . A similar description applies to ${}^0\mathbf{E}(B)$, and in these terms the homomorphism ${}^0\mathbf{E}(\varphi)$ coincides with φ on A and is the identity on X . Thus, ${}^1\mathbf{E}(\varphi) = \mathbf{H}({}^0\mathbf{E}(\varphi)) = \mathbf{H}(\varphi)[X]$ is an isomorphism. As the spectral sequences converge to $\mathbf{H}(A[X])$ and $\mathbf{H}(B[X])$, respectively, it follows that $\mathbf{H}(\varphi[X])$ is an isomorphism, as desired.

(b) It suffices to prove that η is a quasi-isomorphism. Obvious direct limit considerations yield a reduction to the case of finite Λ , and then the Künneth formula provides a further reduction to the case when Λ contains a single element λ . Thus, we only have to show that $\mathbf{H}(k[y_\lambda, \tilde{y}_\lambda]) \cong k$, which is trivial. \square

For the rest of this section R denotes a finitely generated commutative graded k -algebra. We give in detail the construction of one of the minimal models from [2].

(1.8) *Minimal models.* Choose a *minimal* set $\{t_1, \dots, t_e\}$ of homogeneous generators of the graded k -algebra R and a set of variables $X_0 = \{x_1, \dots, x_e\}$ with x_i assigned external degree 0 and the same internal degree as t_i . The graded polynomial ring $P = k[X_0]$ then maps onto R by a homomorphism of k -algebras which sends x_i to t_i . Clearly, the kernel I of this homomorphism is contained in \mathfrak{n}^2 , where $\mathfrak{n} = (X_0)P$.

Choose a *minimal* system of homogeneous generators $\mathbf{f} = \{f_1, \dots, f_r\}$ of I , and adjoin to P a set X_1 of variables of external degree 1, with $\partial(X_1) = \mathbf{f}$. The DG algebra $k[X_{\leq 1}] = k[X_0, X_1]$ has $\mathbf{H}_0(k[X_{\leq 1}]) = R$. As $k[X_{\leq 1}]$ is the usual Koszul complex over P on the sequence \mathbf{f} , it has no homology in positive external degrees if R is a complete intersection, and in this case the construction is complete.

If I is not generated by a regular sequence, then adjoin a set of variables X_2 of external degree 2 and appropriate internal degrees to kill a set of homogeneous cycles whose classes provide a minimal set of generators of the internally graded R -module $\mathbf{H}_1(k[X_{\leq 1}])$. The DG algebra $k[X_{\leq 2}] = k[X_{\leq 1}][X_2]$ obtained at this stage has the desired homology in degrees 0 and 1. Proceeding in this way, by *adjoining at the $(i+1)$ st step a set of variables X_{i+1} of external degree $i+1$ to kill homogeneous cycles which represent a minimal set of generators of $\mathbf{H}_i(k[X_{\leq i}])$* , we get a quasi-isomorphism $k[X] \rightarrow R$ such that

$$(1.9) \quad \partial(X_1) = I \subseteq \mathfrak{n}^2 \quad \text{and} \quad \partial(X_{i+1}) \subset \mathfrak{n}X_i + k[X_{< i}]_i \quad \text{for } i \geq 0.$$

A DG k -algebra $k[X]$ with these properties is easily shown to be unique up to an isomorphism which induces the identity on R . Slightly modifying the terminology of [2], we call it a *minimal model* of the graded k -algebra R .

We use it in the following special case of a result of [4].

(1.10) **Proposition.** *Let $k[X]$ be a minimal model of the k -algebra R , and let $x \mapsto \tilde{x}$ be a bijection to a set of new indeterminates \tilde{X} , such that $|\tilde{x}| = |x| + 1$. There is then a free extension $\iota: k[X] \hookrightarrow k[X, \tilde{X}]$ such that*

$$\partial(\tilde{x}) - x \in \mathfrak{n}\tilde{X}_i + ((X_{<i})k[X_{<i}, \tilde{X}_{<i}])_i \text{ for } x \in X_i \text{ with } i \geq 0,$$

and such that the canonical projection $\varkappa: k[X, \tilde{X}] \rightarrow k$ is a quasi-isomorphism.

Proof. We set $k[X_{\geq i}] = k[X]/(X_{<i})k[X]$, and denote by \tilde{X}_i the elements of \tilde{X} of external degree i .

Define $k[X, \tilde{X}_1]$ to be the free extension of $k[X]$ with $\partial(\tilde{x}) = x$ for $\tilde{x} \in \tilde{X}_1$. The ideal generated by X_0 and \tilde{X}_1 in $k[X_0, \tilde{X}_1]$ is stable for the differential, and (1.7.b) shows it has trivial homology. As $k[X, \tilde{X}_1]^\natural$ is free over $k[X_0, \tilde{X}_1]^\natural$, the projection $k[X, \tilde{X}_1] \rightarrow k[X, \tilde{X}_1]/(X_0, \tilde{X}_1)k[X, \tilde{X}_1] = k[X_{\geq 1}]$ is a quasi-isomorphism by (1.5).

Let now q be a positive integer, and assume by induction that a free extension $k[X] \hookrightarrow k[X, \tilde{X}_{\leq q}]$ has been constructed, such that for $0 \leq i < q$ one has

$$(1.11) \quad \partial(\tilde{x}) - x \in k[X_{<i}, \tilde{X}_{\leq i}]_i \text{ for } x \in X_i,$$

and for $1 \leq i \leq q$ the canonical projection

$$\varkappa^{[i]}: k[X, \tilde{X}_{\leq i}] \rightarrow k[X, \tilde{X}_{\leq i}]/(X_{<i}, \tilde{X}_{\leq i})k[X, \tilde{X}_{\leq i}] = k[X_{\geq i}]$$

is a quasi-isomorphism. We see from (1.9) that $\partial(X_q) = 0$ in $k[X_{\geq q}]$, hence $H_q(k[X_{\geq q}]) = kX_q$. Each $x \in X_q$ lifts through $\varkappa^{[q]}$ to a homogeneous cycle z_x such that

$$z_x - x - \sum_{y \in X_q} a_y y \in k[X_{<q}, \tilde{X}_{\leq q}]_q$$

for suitable $a_y \in \mathfrak{n}$. We have $\mathfrak{n} = \partial(k[X_0]\tilde{X}_1)$, hence for each $y \in X_q$ there is a $b_y \in k[X_0]\tilde{X}_1$ with $a_y = \partial(b_y)$. As $q \geq 1$, the cycle z_x is homologous in $k[X, \tilde{X}_{\leq q}]$ with the cycle $z_x - \partial(\sum_{y \in X_q} b_y y)$. Thus, we may assume that in the lifting above $a_y = 0$ for $y \in X_q$. The free extension $k[X, \tilde{X}_{\leq q}] \hookrightarrow k[X, \tilde{X}_{\leq q+1}]$ with $\partial(\tilde{x}) = z_x$ for $x \in X_q$ then yields a free extension of $k[X]$ in which the relation (1.11) holds for $0 \leq i < q+1$.

Next we decompose $\varkappa^{[q+1]}$ in the form

$$k[X, \tilde{X}_{\leq q+1}] \rightarrow k[X_{\geq q}, \tilde{X}_{q+1}] \rightarrow k[X_{\geq q+1}].$$

The first map is the extension $\varkappa^{[q]}[\tilde{X}_{q+1}]$ of $\varkappa^{[q]}$, cf. (1.6). As $\varkappa^{[q]}$ is a quasi-isomorphism by the induction hypothesis, its extension has the same property due to (1.7.a). Since we have already established (1.11) for $i = q$, we see that $\partial(\tilde{x}) = x$ for $x \in X_q$ in the DG algebra $k[X_q, \tilde{X}_{q+1}]$. Thus, in this DG algebra the set $X_q \cup \tilde{X}_{q+1}$ generates a DG ideal, whose homology is trivial by (1.7.b). The second map above is the factorization from $k[X_{\geq q}, \tilde{X}_{q+1}]$ of the DG ideal generated by $X_q \cup \tilde{X}_{q+1}$. As $k[X_{\geq q}, \tilde{X}_{q+1}]^{\natural}$ is free over $k[X_q, \tilde{X}_{q+1}]^{\natural}$, it follows from (1.5) that this map is a quasi-isomorphism. We now conclude that the composed map $\varkappa^{[q+1]}$ is a quasi-isomorphism, and the induction step is complete.

The upshot of the preceding discussion is that $\varkappa: k[X, \tilde{X}] \rightarrow k$ is a quasi-isomorphism, and that (1.11) holds for all $i \geq 0$. Furthermore, by construction $\partial(X_1)$ is a homogeneous minimal set of generators of \mathfrak{n} , and for $i \geq 1$ the homogeneous set of cycles $\partial(\tilde{X}_{i+1})$ represents a basis of the k -space $H_i(k[X, \tilde{X}_{\leq i}])$. Gulliksen [10, (6.2)] proves that an extension formed in this way satisfies

$$(1.12) \quad \partial(k[X, \tilde{X}]_{i+1}) \subseteq ((X)k[X, \tilde{X}])_i = \mathfrak{n}\tilde{X}_i + ((X_{\leq i})k[X_{\leq i}, \tilde{X}_{< i}])_i$$

Combining this with (1.11) we obtain:

$$\begin{aligned} \partial(\tilde{x}) - x &\in \left(k[X_{< i}, \tilde{X}_{\leq i}]_i\right) \cap \left(\mathfrak{n}\tilde{X}_i + ((X_{\leq i})k[X_{\leq i}, \tilde{X}_{< i}])_i\right) \\ &= \mathfrak{n}\tilde{X}_i + ((X_{< i})k[X_{< i}, \tilde{X}_{< i}])_i, \end{aligned}$$

as desired. □

(1.13) *Derivations and differentials.* Let A be a DG k -algebra and F be a DG A -module. A homomorphism of k -spaces $\theta: A \rightarrow F$ of external degree $|\theta|$ (and homogeneous for the internal gradings) is called a (k -) *derivation* if for all a, b in A there is an identity

$$\theta(ab) = (-1)^{(|a|+|\theta|)|b|} b\theta(a) + (-1)^{|a||\theta|} a\theta(b).$$

(The reader is invited to verify that for $F = A$ this gives the usual notion of (skew-) derivation.) The set $\text{Der}_k(A, F)$ of all k -derivations is a DG A -module with operation and differential induced from the inclusion $\text{Der}_k(A, F) \subseteq \text{Hom}_k(A, F)$, to wit: $(a\theta)(b) = a\theta(b)$ and differential $\partial(\theta) = \partial \circ \theta - (-1)^{|\theta|} \theta \circ \partial$.

The usual formalism of algebraic analysis [12] carries over smoothly, in particular one sees that the functor $\text{Der}_k(A, -)$ is representable by $\Omega_{A|k} = J/J^2$, where J is the kernel of the multiplication $A \otimes A \rightarrow A$, $a \otimes b \mapsto ab$. The fact that J is a DG ideal provides $\Omega_{A|k}$ with a canonical structure of DG A -module, called the *DG module of differentials* of the DG k -algebra A . The k -linear chain map $d_{A|k}: A \rightarrow \Omega_{A|k}$, which is homogeneous of internal and external degree zero, is a universal derivation, and the functorial isomorphism

$$\text{Hom}_A(\Omega_{A|k}, F) \cong \text{Der}_k(A, F), \quad \gamma \mapsto \gamma \circ d_{A|k},$$

is one of DG A -modules. It follows that γ is a morphism of DG A -modules if and only if the corresponding derivation θ is of external degree zero and satisfies $\partial\theta = \theta\partial$.

The definition of the Euler derivation of a graded k -algebra R may be stretched to define a derivation of any DG algebra A . Namely, when an element $a \in A_i$ is written as $a = \sum_{n \in \mathbb{Z}} a_n$ with $a_n \in A_i$ homogeneous of internal degree n , consider the map $a \mapsto \sum_{n \in \mathbb{Z}} n a_n$. One checks immediately that this is a k -derivation of A which preserves both external and internal degrees and commutes with the differential. By the preceding remarks, it defines an *Euler morphism* $\omega: \Omega_{A|k} \rightarrow A$.

(1.14) *Example.* When $k \hookrightarrow k[X]$ is a free extension, these structures acquire the following explicit form: The $k[X]^{\natural}$ -module $\Omega_{k[X]|k}^{\natural}$ is free on a basis $\{dx\}_{x \in X}$, where $d = d_{k[X]|k}$; the differential of $\Omega_{k[X]|k}$ is defined by $\partial(dx) = d(\partial x)$; its Euler morphism is given by $\omega(dx) = n(x)x$, where $n(x)$ is the internal degree of x (recall from (1.6) that in a free extension each variable is homogeneous for the internal degree).

With the notation of (1.10) and (1.14), we now have:

(1.15) **Lemma.** *There is a degree 1 homomorphism $\chi: \Omega_{k[X]|k} \rightarrow k[X, \tilde{X}]$ of DG modules over $k[X]$, such that $\partial\chi + \chi\partial = \omega$ and*

$$\chi(dx) - n(x)\tilde{x} \in \mathfrak{n}\tilde{X}_{i+1} + k[X_{\leq i+1}, \tilde{X}_{\leq i}]_{i+1} \text{ for } x \in X_i \text{ and } i \geq 0.$$

Proof. We construct the desired homomorphism by induction on i , extending a degree 1 homomorphism $\prod_{|x| \leq i} k[X]dx \rightarrow k[X, \tilde{X}]$ to the larger DG submodule $\prod_{|x| \leq i+1} k[X]dx$. By abuse of notation, we denote any of these maps by χ .

Let $\chi: \prod_{|x|=0} k[X]dx \rightarrow k[X, \tilde{X}]$ be the homomorphism of DG $k[X]$ -modules defined by $\chi(dx) = n(x)\tilde{x}$ for $x \in X_0$. Clearly, it satisfies

$$(1.16) \quad \partial\chi(dx) = n(x)x.$$

Assume by induction that χ with the required properties has been constructed on $\prod_{|x| < i} k[X]dx$. For $x \in X_i$ consider the element $\omega(dx) - n(x)\partial(\tilde{x}) - \chi\partial(dx) \in k[X_{\leq i}, \tilde{X}_{\leq i}]$. An easy computation which uses the induction hypothesis shows it is a cycle. Since $i \geq 1$, it is even a boundary, which we choose to write in the form $\partial(v_x + w_x)$ with $v_x \in P\tilde{X}_{i+1}$, where $P = k[X_0]$, and $w_x \in k[X_{\leq i+1}, \tilde{X}_{\leq i}]_{i+1}$. As d is a derivation, (1.9) shows that:

$$d(\partial X) \subseteq d\left(\left((X)k[X]\right)^2\right) \subseteq (X)d(k[X]) = (X)\Omega_{k[X]|k}.$$

Since χ is a homomorphism of DG $k[X]$ -modules, for $i \geq 0$ this implies:

$$\chi\partial(dX_i) = \chi d(\partial X_i) \subseteq \chi\left(\left((X)\Omega_{k[X]|k}\right)_{i-1}\right) \subseteq \left((X)k[X, \tilde{X}]\right)_i \subseteq \mathfrak{n}X_i + k[X_{< i}, \tilde{X}_{\leq i}]_i.$$

Furthermore, it follows from (1.9) that $\partial(w_x) \in \mathfrak{n}X_i + k[X_{< i}, \tilde{X}_{\leq i}]_i$. Applying these observations in conjunction with (1.10) we see that

$$\partial(v_x) = \omega(dx) - n(x)\partial(\tilde{x}) - \chi\partial(dx) - \partial(w_x) \in \mathfrak{n}X_i + k[X_{< i}, \tilde{X}_{\leq i}]_i.$$

From (1.10) we now conclude that $v_x \in \mathfrak{n}\tilde{X}_{i+1}$.

We extend χ to $\prod_{|x| \leq i} k[X]dx$ by setting $\chi(dx) = n(x)\tilde{x} + v_x + w_x$. This gives a homomorphism of DG $k[X]$ -modules, such that the desired relation for $\chi(dx) - n(x)\partial(\tilde{x})$ and the equality $\partial\chi(dx) + \chi\partial(dx) = n(x)\partial(\tilde{x}) + \partial(v_x + w_x) + \chi\partial(dx) = \iota\omega(dx)$ both hold for $x \in X_{\leq i}$. This completes the induction step, and the proof of the lemma. \square

(1.17) *Minimal cotangent complexes.* Let $k[X] \rightarrow R$ be a minimal model of the graded k -algebra R , as in (1.8). We set $L_{R|k} = R \otimes_{k[X]} \Omega_{k[X]|k}$ and call this complex of graded R -modules a *minimal cotangent complex* of the k -algebra R . Its i th module L_i is free with basis dX_i . The differential $\partial_1: L_1 \rightarrow L_0$ is given by the jacobian matrix $(\frac{\partial f_j}{\partial x_i})$, hence

$$(1.18) \quad H_0(L_{R|k}) = \Omega_{R|k}.$$

If $k[X'] \rightarrow R$ also is a minimal model, then as noted after (1.9) there is an isomorphism $\varphi: k[X] \cong k[X']$ of DG k -algebras such that $H_0(\varphi)$ induces the identity on R . This induces a φ -equivariant isomorphism $\gamma: \Omega_{k[X]|k} \cong \Omega_{k[X']|k}$ of DG modules. Thus, there is an isomorphism $\gamma: R \otimes_{k[X]} \Omega_{k[X]|k} \cong R \otimes_{k[X']} \Omega_{k[X']|k}$ which shows that $L_{R|k}$ is unique up to isomorphism of complexes of graded R -modules.

Finally, we remark that since the characteristic of k is zero, $L_{R|k}$ is a cotangent complex of the k -algebra R in the sense of the theory of André and Quillen by [17, (9.5)].

(1.19) *Proof of the Main Theorem.* Let $k[X, \tilde{X}]$ be the DG algebra constructed in (1.10). It follows from (1.12) that $R \otimes_{k[X]} k[X, \tilde{X}]$ is a minimal complex of graded free R -modules. By (1.5) its homology is trivial except in external degree 0, where it is isomorphic to k . Thus, we get the expression

$$(1.20) \quad F(k) = R \otimes_{k[X]} k[X, \tilde{X}] = R[\tilde{X}].$$

for the minimal free resolution of the graded R -module k .

Consider next the homomorphisms of DG $k[X]$ -modules $\Omega_{k[X]|k} \xrightarrow{\omega} k[X] \xrightarrow{\iota} k[X, \tilde{X}]$ defined in (1.14) and (1.10). Tensoring them over $k[X]$ with R we get homomorphisms of complexes of graded free R -modules: $L_{R|k} \xrightarrow{\bar{\omega}} R \xrightarrow{\bar{\iota}} F(k) = F$.

Let $\bar{\chi} = R \otimes_{k[X]} \chi: L_{R|k} \rightarrow F$ be the degree 1 homomorphism of complexes of R -modules induced by the homomorphism of DG $k[X]$ -modules χ from (1.15). It satisfies $\bar{\chi}\partial + \partial\bar{\chi} = \bar{\iota}\bar{\omega}$. Remark that the image of $\bar{\iota}\bar{\omega}$ is contained in F_0 , and thus $\alpha = \{\bar{\chi}_i: (L_{R|k})_i \rightarrow F_{i+1}\}_{i \geq 0}$ is a morphism of complexes $L_{R|k} \rightarrow F_{[1]}$, where $F_{[1]}$ is the first truncation of F , as in (1.2).

Note that $F_{[1]}$ is nothing but the minimal free resolution $F(\mathfrak{m})$ of \mathfrak{m} , and that (1.16) implies $H_0(\alpha)$ is equal to the Euler homomorphism $\omega: \Omega_{R|k} \rightarrow \mathfrak{m}$. Therefore, to finish the proof it remains to show $\alpha \otimes_R k$ is injective. Now (1.15) yields

$$(\alpha \otimes k)(dx \otimes 1) \equiv n(x)\tilde{x} \otimes 1 \pmod{k[\tilde{X}_{\leq i}]_{i+1}} \text{ for } x \in X_i \text{ and } i \geq 0,$$

and this clearly provides the desired conclusion. \square

2. COMPLETE INTERSECTIONS

This section contains the proofs of Theorems 1 through 4 announced in the introduction. To deduce the Theorem 4 from the Main Theorem we use the following simple observation.

(2.1) **Lemma.** *Let L be a minimal complex of graded free R -modules, let $\varepsilon: L \rightarrow M$ be an augmentation to a graded R -module M , and let $\mu: M \rightarrow N$ be a homomorphism of graded R -modules. If the augmentation $\mu\varepsilon: L \rightarrow N$ is essential, then so is the augmentation ε .*

Proof. If $\alpha: L \rightarrow F(M)$ is a chain map which extends ε , and $\beta: F(M) \rightarrow F(N)$ is a chain map such that $H_0(\beta) = \mu$, then $\gamma = \beta\alpha: L \rightarrow F(N)$ extends the augmentation $\mu\varepsilon$. Thus, the homomorphism $\alpha \otimes_R k: L \otimes_R k \rightarrow F(M) \otimes_R k$ has to be injective if $\gamma \otimes_R k: L \otimes_R k \rightarrow F(N) \otimes_R k$ has this property. \square

(2.2) *Proof of Theorem 4.* We write L for the minimal cotangent complex $L_{R|k}$. By the Main Theorem the composition of the canonical augmentation $L \rightarrow H_0(L) = \Omega_{R|k}$ with the Euler derivation $\omega: \Omega_{R|k} \rightarrow \mathfrak{m}$ is an essential augmentation. By the lemma, so is the canonical augmentation itself. To obtain the corresponding conclusion for the augmentation of L to $\Omega_{R|k}/t(\Omega_{R|k})$, note that ω factors through the natural projection $\Omega_{R|k} \rightarrow \Omega_{R|k}/t(\Omega_{R|k})$, and again refer to the lemma. This establishes the desired result for $L_{[0]} = L$.

Recall from the introduction that the i th cotangent module C_i of the k -algebra R is defined to be the cokernel of the i th differential of L . If $F = F(\mathfrak{m})$ is the minimal free resolution of \mathfrak{m} , and $\alpha: L \rightarrow F$ is a morphism of complexes of R -modules with $H_0(\alpha) = \omega$, then it defines for each $i \geq 0$ a morphism of complexes $\alpha_{[i]}: L_{[i]} \rightarrow F_{[i]}$. By the first part of the proof, the induced homomorphism of externally graded k -spaces $\alpha_{[i]} \otimes_R k: L_{[i]} \otimes_R k \rightarrow F_{[i]} \otimes_R k$ is injective for $i \geq 0$. Since the homomorphism of graded R -modules $H_0(\alpha_{[i]})$ factors as $H_0(L_{[i]}) = C_i \rightarrow C_i/t(C_i) \rightarrow H_0(F_{[i]})$, this implies that the canonical augmentations of $L_{[i]}$ to C_i and to $C_i/t(C_i)$ are essential. \square

The result which we have just finished proving has the following numerical consequence:

(2.3) **Corollary.** *Let $\epsilon_n(R)$ be the number of degree n variables in a minimal model of the k -algebra R . For each pair of non-negative integers i and j there are inequalities $\epsilon_{i+j}(R) \leq b_j(C_i)$ and $\epsilon_{i+j}(R) \leq b_j(C_i/t(C_i))$.* \square

The construction (1.8) of the minimal model shows that R is a polynomial ring if and only if $\epsilon_n(R) = 0$ for $n \geq 1$, and a complete intersection if and only if $\epsilon_n(R) = 0$ for $n \geq 2$: This explains why the non-negative integers $\epsilon_n(R)$ are known as the *deviations* of R . They also appear in the study of (the singularity of) the ring R through a different avenue. Namely, the Poincaré series of k factors as a formal product

$$P_k(t) = \prod_{j=0}^{\infty} \frac{(1 + t^{2j+1})^{e_{2j+1}}}{(1 - t^{2j+2})^{e_{2j+2}}},$$

which uniquely determines the integers $e_n = e_n(R)$ for $n \geq 1$. The expression $F(k) = R[\tilde{X}]$ obtained in (1.20) identifies $e_n(R)$ with the number of degree n variables in the

Tate resolution [10] of the residue field of k of R . On the other hand, (1.10) shows that $e_{n+1}(R) = \epsilon_n(R)$ for $n \geq 0$.

Gulliksen [9] has proved that if $e_n(R)$ vanishes for $n \gg 0$, then R is a complete intersection. Combining this result with (2.3) we get:

(2.4) **Theorem.** *Let R be a graded k -algebra, let i be a non-negative integer, and let C stand either for a cotangent module C_i of R or for a quotient $C_i/t(C_i)$. If the projective dimension of such a module C is finite, then R is a complete intersection. \square*

To continue we need to identify the cotangent modules C_i in terms of invariants of the minimal model. This is done in the following proposition, cf. [11].

(2.5) **Proposition.** *Let $k[X] \rightarrow R$ be a minimal model of the graded k -algebra R , so that in particular $R \cong P/I$, with $P = k[X_0]$ a graded polynomial algebra. There are then canonical isomorphisms:*

$$C_i \cong \begin{cases} \Omega_{R|k} & \text{for } i = 0; \\ I/I^2 & \text{for } i = 1; \\ H_{i-1}(k[X_{\leq i-1}]) & \text{for } i \geq 2. \end{cases}$$

Proof. Set $J = \text{Ker}(k[X] \rightarrow R)$ and consider the morphism of complexes of R -modules $\delta: J/J^2 \rightarrow R \otimes_{k[X]} \Omega_{k[X]|k} = L_{R|k}$, $a + J^2 \mapsto 1 \otimes da$. For $i \geq 1$, the R -module $(J/J^2)_i$ is free on a basis $\{x + J^2\}_{x \in X_i}$, and δ maps this basis bijectively onto $\{dx\}_{x \in X_i}$. By (1.17) the latter is a basis of $(L_{R|k})_i$ over R , hence δ_i is an isomorphism, and thus $C_i \cong \text{Coker } \partial_{i+1}: (J/J^2)_{i+1} \rightarrow (J/J^2)_i$ for $i \geq 1$.

Denote by $K(j)$ the complex of graded P -modules $k[X_{\leq j}]/k[X_{\leq j-1}]$. In view of the equality $H_i(k[X_{\leq i+1}]) = 0$ for $i \geq 1$, the short exact sequence

$$0 \rightarrow k[X_{\leq i}] \rightarrow k[X_{\leq i+1}] \rightarrow K(i+1) \rightarrow 0$$

produces for each i an exact sequence

$$(2.6) \quad H_{i+1}(k[X_{\leq i+1}]) \xrightarrow{\eta_{i+1}} H_{i+1}(K(i+1)) \xrightarrow{\partial_{i+1}} H_i(k[X_{\leq i}]) \rightarrow 0.$$

Remarking that $K(j)_{j-1} = 0$, $K(j)_j = PX_j$, and $\partial(K(j)_{j+1}) = IX_j$ by (1.9), we see that $H_j(K(j)) = RX_j = (J/J^2)_j$ for $j \geq 1$. An easy computation establishes that under these isomorphisms the differential ∂_{i+1} of J/J^2 corresponds to $\eta_i \partial_{i+1}$.

A look at (2.6) shows that for $i \geq 1$ the homomorphism ∂_{i+1} is surjective, and thus $C_i \cong \text{Coker}(\eta_i \partial_{i+1}) = \text{Coker } \eta_i$. A second look at (2.6) now yields for $i \geq 2$ an isomorphism $\text{Coker } \eta_i \cong H_{i-1}(k[X_{\leq i-1}])$, as desired. The expression for C_1 is obtained by computing $\text{Coker } \eta_1$ directly, while that for C_0 has been observed in (1.18). \square

As an immediate consequence of the last two results we have:

(2.7) **Corollary.** *Let C denote either the module of differentials $\Omega_{R|k}$, or the conormal module I/I^2 (for some regular embedding $R \cong P/I$), or the first homology module $H_1(K)$ of the Koszul complex K on a homogeneous system of generators of I , or the quotient modulo torsion of any of these modules. If $\text{projdim}_R C$ is finite, then R is a complete intersection.* \square

The corollary proves that (iii) implies (i) in Theorem 3, and opens the way to:

(2.8) *Proof of Theorem 1.* We only have to prove that (iii) implies (i), and we have just seen in (2.7) that R is a complete intersection. In order to show it is reduced, pick a minimal prime ideal \mathfrak{p} in R . The module $(\Omega_{R|k})_{\mathfrak{p}} = \Omega_{R_{\mathfrak{p}}|k}$ is then finite and of finite projective dimension over the artinian ring $R_{\mathfrak{p}}$, hence is free. By the jacobian criterion [12, (7.4.b)] this implies $R_{\mathfrak{p}}$ is regular, and thus a field. \square

Next we give a characterization of normal complete intersections.

(2.9) **Theorem.** *For a graded k -algebra R the following conditions are equivalent:*

- (i) *R is a normal complete intersection;*
- (ii) *$\text{projdim}_R \Omega_{R|k} \leq 1$ and $\Omega_{R|k}$ is torsion-free;*
- (iii) *$\text{projdim}_R \Omega_{R|k}/t(\Omega_{R|k}) < \infty$ and R is reduced.*

Proof. The fact that (i) implies (ii) is contained in [14, (8.2.1)]. Clearly, (ii) contains the first condition in (iii) and implies the second one by Theorem 1.

To get (i) from (iii) note that by (2.7) the ring R is a complete intersection, hence by Serre's criterion it suffices to prove it is regular in codimension one. Let \mathfrak{p} be a height one prime ideal. As R is reduced, the torsion submodule localizes and thus yields $(\Omega_{R|k}/t(\Omega_{R|k}))_{\mathfrak{p}} \cong \Omega_{R_{\mathfrak{p}}|k}/t(\Omega_{R_{\mathfrak{p}}|k})$. The latter module, being both torsion-free and of finite projective dimension over the one-dimensional Cohen–Macaulay ring $R_{\mathfrak{p}}$, is actually free. By Lipman [15] this implies $R_{\mathfrak{p}}$ is regular, so we have won. \square

To validate all the results announced in the introduction it only remains to present:

(2.10) *Proof of Theorem 2 and Theorem 3.* By [3, (4.1)] the Betti numbers of each finite module over a complete intersection have the property described in condition (ii) of Theorem 2. As in both theorems the implications (ii) \Rightarrow (iii) \Rightarrow (iv) are clear, we only need to show that (iv) implies (i).

Let C_i be the i th cotangent module of the k -algebra R . For each $i \geq 0$, the inequalities established in (2.3) combine to show that the formal power series $\sum_{j=0}^{\infty} \epsilon_{i+j} t^j$ is a coefficientwise lower bound both for the Poincaré series of C_i and for that of its quotient modulo torsion. Thus, (2.5) shows that in either version of Theorem 2 and Theorem 3, condition (iv) implies that the formal power series $E(t) = \sum_{j=0}^{\infty} \epsilon_j t^j$ converges in the open unit disk.

If R is not a complete intersection, then by [9] the series $E(t)$ is infinite, hence by [6] its radius of convergence is equal to the radius of convergence $\rho(k)$ of the Poincaré series $P_k(t)$ of the R -module k , and $\rho(k) < 1$ by [7, (IV.7)]. \square

3. LOCAL RINGS

Here we briefly discuss possible extensions of the results of this paper to the case when I is an ideal of finite projective dimension in a noetherian local ring P .

A well known result of Ferrand [8] and Vasconcelos [20] shows that if I/I^2 is free over $R = P/I$, then I is generated by a regular sequence. Subsequent work has led Vasconcelos to conjecture [21, Conjecture \mathfrak{C}_1] that the same conclusion holds when the condition on I/I^2 is relaxed to the assumption that it has finite projective dimension over R . He and others have proved this is indeed the case when $\text{projdim}_R I/I^2 \leq 1$, when I has small codimension, or when it is an almost complete intersection, cf. [22] for further references.

Let P be an essentially of finite type, regular algebra over a field k of characteristic zero. In this case Ferrand [8] has proved that when R is reduced it is a complete intersection if and only if $\text{projdim}_R \Omega_{R|k} \leq 1$, and Vasconcelos [21, Conjecture \mathfrak{C}_2] has conjectured that if $\text{projdim}_R \Omega_{R|k}$ is finite then R is a complete intersection. This is known to hold for the same I as above, cf. [22] and the literature cited there.

Our Theorems 1 and 3 contain proofs of these conjectures for a localization at the irrelevant maximal ideal of a graded algebra over a field of characteristic zero. Furthermore, the statements of all the results of the introduction, with the exception of the Main Theorem, are meaningful under more general assumptions, so the problem arises if they remain valid in the framework of the original conjectures.

In connection with conjecture \mathfrak{C}_1 it is interesting to investigate whether the canonical augmentation $J/J^2 \rightarrow I/I^2$ is essential, where $J = \text{Ker}(P[X] \rightarrow R)$ with $P[X]$ a minimal model of the P -algebra R . In connection with \mathfrak{C}_2 the same question stands for the canonical augmentation $L_{R|k} \rightarrow \Omega_{R|k}$. The pertinent growth conditions on the ranks of the free modules of those complexes are available in this generality [4].

However, it is not possible to extend Theorem 4 to the local setup in the strongest possible formulation. Indeed, Platte [16] has constructed a local ring R which is the localization of an affine domain over \mathbb{C} , and whose module of differentials has a torsion minimal generator. It follows that the canonical augmentation $L_{R|\mathbb{C}} \rightarrow \Omega_{R|\mathbb{C}}/t(\Omega_{R|\mathbb{C}})$ is not essential.

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