HOCHSCHILD HOMOLOGY CRITERIA FOR SMOOTHNESS

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Introduction

When R is a commutative algebra over a field k, its *Hochschild homology* can be defined by the formula

$$\mathrm{HH}_*(R) = \mathrm{Tor}_*^{R\otimes R}(R, R),$$

where the tensor product is over k, and R is viewed as an $R \otimes R$-module via the multiplication map $\mu : R \otimes R \to R$, $\mu(a' \otimes a'') = a'a''$. The $\pitchfork$-product of Cartan-Eilenberg, cf. [CE, Chapter XI], provides $\mathrm{HH}_*(R)$ with a structure of skew-commutative (also called graded-commutative) graded algebra over $\mathrm{HH}_0(R) = R$. As $\mathrm{HH}_1(R)$ is canonically identified with $\Omega^1_{R|k}$, the module of Kähler differentials of the k-algebra R, there is a natural homomorphism of graded R-algebras:

$$\gamma^*_{R|k} : \Omega^*_{R|k} = \Lambda^*_R \Omega^1_{R|k} \to \mathrm{HH}_*(R).$$

A finitely generated k-algebra is said to be *smooth* (or: *geometrically regular*) if for any finite field extension k' of k the ring $R \otimes_k k'$ is regular. The Hochschild homology of such algebras is computed by a celebrated result.

Theorem of Hochschild, Kostant, and Rosenberg, [HKR, (5.2)]. *If a k-algebra R is finitely generated and smooth, then the R-module $\Omega^1_{R|k}$ is finitely generated projective, and the map $\gamma^*_{R|k}$ is an isomorphism.*

In particular, the Hochschild homology of a smooth algebra vanishes in sufficiently high degrees. The purpose of this paper is to establish a number of converses.

1. The results

Our strongest Hochschild criterion for smoothness reads as follows.

Theorem. *Let R be a finitely generated algebra over a field k.*

If $\mathrm{HH}_i(R) = 0 = \mathrm{HH}_j(R)$ for some positive even integer i and some positive odd integer j, then R is smooth.

The proof, given in the next section, uses arguments from differential graded homological algebra, and intuition from the "dictionaries" between the homology of local rings and that of loop spaces, cf. [Av, AH, FHJLT], respectively between Hochschild homology and the homology of free loop spaces, cf. [Go, BV, HV].

As an immediate consequence of the preceding theorems, we obtain various characterizations of smoothness in terms of Hochschild homology.

L.L.A. was partly supported by NSF Grant No. DMS-9102951.

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Corollary. *For an algebra R which is finitely generated over a field k, the following conditions are equivalent.*

(i) *R is smooth over k.*

(ii) *$\gamma^*_{R|k}$ is an isomorphism.*

(iii) *The R-algebra $\mathrm{HH}_*(R)$ is generated by its elements of degree 1.*

(iv) *There is an integer n such that $\mathrm{HH}_i(R) = 0$ for $i > n$.*

Proof. (i) implies (ii) by the Hochschild-Kostant-Rosenberg theorem; (ii) $\implies$ (iii) is obvious; (iii) $\implies$ (iv) as the R-algebra $\mathrm{HH}_*(R)$ is skew-commutative, and the R-module $\mathrm{HH}_1(R) = \Omega^1_{R|k}$ is finitely generated; (iv) implies (i) by the Theorem. □

REMARK 1. The implication (ii) $\implies$ (i) shows that only one of the properties of Hochschild homology established by the Hochschild-Kostant-Rosenberg theorem is sufficient for the geometric regularity of R. The relation of the other property—the projectivity of $\Omega^1_{R|k}$—to geometric regularity is well known by the classical Jacobian criterion for smoothness: it is sufficient when the characteristic of k is zero, cf. [Ku, (7.2)], but in general it has to be replaced by the stronger requirement that for any $\mathfrak{p} \in \operatorname{Spec} R$ the $R_\mathfrak{p}$-module $(\Omega^1_{R|k})_\mathfrak{p}$ is free with rank equal to $\dim R_\mathfrak{p} + \dim R/\mathfrak{p}$, cf. [Ku, (7.14)].

REMARK 2. Use of the cotangent homology theory of commutative algebras, constructed by André and Quillen, shows that the projectivity of the module of Kähler differentials may be complemented with only a small piece of the Hochschild structure. Indeed, it is implicit in [Qu, Sections 5 and 6] and explicit in [Rn] that R is geometrically regular over k if (and only if) $\Omega^1_{R|k}$ is projective and $\gamma^2_{R|k} : \Omega^2_{R|k} \to \mathrm{HH}_2(R)$ is surjective.

REMARK 3. The validity of the implication (iv) $\implies$ (i) has been conjectured by Rodicio [Rd] in the special case of a field k of characteristic zero. The conjecture, which has provided the motivation for this paper, is proved in [MR] for k-algebras R which are locally complete intersection, and the assertion of the theorem above is later established for such algebras in [LR].

REMARK 4. When the present paper was being prepared for publication we learned that A. Campillo, J. A. Guccione, J. J. Guccione, M. J. Redondo, A. Solotar, and O. E. Villamayor have independently proved the theorem above under the additional assumption that $\operatorname{char}(k) = 0$, thus settling the original conjecture of Rodicio. Their methods, cf. [BACH], are close to ours, as they also use techniques of minimal models inspired by rational homotopy theory.

2. The proofs

We shall establish the theorem from the preceding section in the following slightly more precise converse form.

Theorem 1. *Let R be an algebra generated by m elements over a field k.*

If R is not smooth, then there is an integer $n \le m$ such that $\mathrm{HH}_i(R) \neq 0$ for $i \le n$ and for $i = n + 2j$ with $j > 0$.

Proof. We make the convention that all unlabeled tensor products are over k. Let $\overline{k}$ be an algebraic closure of k. For the $\overline{k}$-algebra $R \otimes \overline{k}$ we have an isomorphism

$$\mathrm{HH}_*(R \otimes \overline{k}) \cong \mathrm{HH}_*(R) \otimes \overline{k}$$

by a classical property of Hochschild homology. On the other hand, an easy argument by faithfully flat descent shows that $R \otimes_k \overline{k}$ is not geometrically regular over $\overline{k}$. Thus, for the rest of the proof we may assume that k is algebraically closed.

Under this condition geometric regularity is equivalent to regularity, hence there is a maximal ideal $\mathfrak{p}$ of R such that the local ring $R_{\mathfrak{p}}$ is not regular. As R is a quotient of a polynomial k-algebra in m variables, $R_{\mathfrak{p}}$ is a quotient of a regular local ring Q, whose maximal ideal $\mathfrak{n}$ is generated by m elements, and whose residue field $Q/\mathfrak{n}$ is—by the Nullstellensatz—isomorphic to k. If $R_{\mathfrak{p}} = Q/I$ and $I \not\subseteq \mathfrak{n}^2$, then taking $f \notin \mathfrak{n}^2$ one sees that $R_{\mathfrak{p}}$ is a quotient of the regular local ring $Q/(f)$, whose maximal ideal $\mathfrak{n}/(f)$ is generated by $m-1$ elements. Iterating this procedure one arrives at a presentation $R_{\mathfrak{p}} = Q/I$, with Q a regular local algebra essentially of finite type over k and I a non-zero ideal contained in the square of the maximal ideal of Q; note that the conditions on I express the fact that $R_{\mathfrak{p}}$ is not regular.

Now, as Hochschild homology commutes with localization, that is, there is an isomorphism

$$\mathrm{HH}_*(R)_{\mathfrak{p}} \cong \mathrm{HH}_*(R_{\mathfrak{p}}),$$

cf. [Br, Proposition 1], the assertion of Theorem 1 is seen to be a consequence of the next result. □

Theorem 2. *Let k be a field, and let Q be a regular local essentially of finite type k-algebra, with maximal ideal $\mathfrak{n}$ minimally generated by n elements, and with residue field $Q/\mathfrak{n}$ isomorphic to k.*

If I is a non-zero ideal contained in $\mathfrak{n}^2$, then the k-algebra $R = Q/I$ satisfies

$$\mathrm{HH}_i(R) \neq 0 \quad \textit{for} \quad i \leq n \quad \textit{and for} \quad i = n + 2j \quad \textit{with} \quad j > 0.$$

Proof. The arguments are based on constructions from DG (= differential graded) homological algebra. If $X = \bigoplus_{j\geq 1} X_j$ is a graded k-vector space, we will consider the free commutative graded algebra on X which is the tensor product of the polynomial algebra on $\bigoplus_{j>0} X_{2j}$ with the exterior algebra on $\bigoplus_{j\geq 0} X_{2j+1}$: it is denoted ΛX in the literature on rational homotopy. If A is a skew-commutative DG k-algebra, we denote by $A[X]$ a DG algebra with underlying graded algebra $A \otimes \Lambda_k X$, and with differential which extends that of A. We set $X_{\leq i} = \oplus_{j=1}^{i} X_j$. If a is a homogeneous element we denote its degree by $|a|$.

The familiar process of adjoining variables of degree i, $i \geq 1$, in order to kill the cycles of degree $i-1$, cf. [Ta, GL] shows that the canonical projection $Q \to R$ can be factored as a composition of homomorphisms of DG algebras $Q \to Q[X] = A \xrightarrow{\pi} R$, where the first map is the natural injection, and

(1) *π is a quasi-isomorphism;*

that is, the map $H(\pi)$ induced in homology is an isomorphism.

Clearly, one can construct X by inductively adjoining to $X_{\leq i-1}$ a vector space sitting in degree i, whose basis kills a *minimal* set of generators of $\mathrm{Ker}(H(Q[X_{\leq i-1}]) \to R)$. As observed first in [Wo], a DG algebra $Q[X]$ is obtained in this manner if and only if its differential ∂ is decomposable:

(2.1) *if $x \in X_1$, then $\partial x \in \mathfrak{n}^2$;*

(2.2) *if $x \in X_i$, with $i > 1$, then $\partial x \in \mathfrak{n} X_{i-1} \mod (X)^2 Q[X]$.*

Note also that the condition $I \neq 0$ translates to

(3) $X_1 \neq 0$.

Consider next the maximal ideal $\mathfrak{m}' = Q \otimes \mathfrak{n} + \mathfrak{n} \otimes Q \subseteq Q \otimes Q$. The multiplication $Q \otimes Q \to Q$ maps $(Q \otimes Q) \backslash \mathfrak{m}'$ to $Q \backslash \mathfrak{n}$, hence it factors into $Q \otimes Q \to P \xrightarrow{\mu} Q$, where the first homomorphism is the localization map to $P = (Q \otimes Q)_{\mathfrak{m}'}$. As Q is the localization of a k-algebra of finite type, the ring P is noetherian and local; we denote its maximal ideal by $\mathfrak{m}$. The homomorphism $Q \to Q \otimes Q$, $a \mapsto a \otimes 1$ induces a flat local homomorphism $Q \to P$, whose fibre $P/\mathfrak{n}P$, is isomorphic to Q. As Q is a regular local ring of dimension n, the ring P is seen to be regular local of dimension $2n$, cf. [Ma, (23.7) and (15.1)]. Thus, by [Ma, (14.2)] we obtain the following.

(4) Ker μ *is generated by a P-regular sequence whose elements are linearly independent modulo* $\mathfrak{m}^2$.

Finally, set $U = (A \otimes A) \otimes_{Q \otimes Q} P$. This skew-commutative differential graded algebra is generated over $U_0 = P$ by two copies of X and we shall also write $U = P[X', X'']$ where $X' = X \otimes 1$, $X'' = 1 \otimes X$.

Now we have a commutative diagram of homomorphisms of DG algebras:

$$\begin{array}{ccccccc} U & = & P[X',X''] & \xleftarrow{\sigma} & A \otimes A & \xrightarrow{\pi \otimes \pi} & R \otimes R \\ & & \rho \downarrow & & \downarrow & & \downarrow \\ & & A & = & A & \xrightarrow{\pi} & R \end{array}$$

in which the unlabeled arrows are multiplication maps, σ is induced by the localization $Q \otimes Q \to P$, and ρ is defined on $U_0 = P$ by $\rho_0 = \mu : P \to Q$, and for $x \in X$ by $\rho(x') = \rho(x'') = x$.

By the functoriality of the DG Tor functor of Eilenberg and Moore, cf. [Mo, AH], we have the homomorphisms of graded R-algebras:

$$\mathrm{Tor}^U(R,A) \xleftarrow{\mathrm{Tor}^{\sigma}(R,A)} \mathrm{Tor}^{A \otimes A}(R,A) \xrightarrow{\mathrm{Tor}^{\pi \otimes \pi}(R,\pi)} \mathrm{Tor}^{R \otimes R}(R,R).$$

Claim. $\mathrm{Tor}^{\pi \otimes \pi}(R, \pi)$ *is an isomorphism.*

Proof of the claim. As $H(\pi)$ is an isomorphism by (1), and then $H(\pi \otimes \pi)$ is an isomorphism by the Künneth theorem, the assertion follows from the Eilenberg-Moore comparison theorem [Mo, (2.3)].

Another way to see this is as follows: The Hochschild homology of a differential k-algebra (A, ∂) is the total homology of the bigraded Hochschild complex associated to (A, ∂) (cf. [Go] for example), and it is easy to see that a quasi-isomorphism of differential graded algebras induces an isomorphism between the homologies of their Hochschild complexes, [Go, (2.9)]. □

Claim: $\mathrm{Tor}^{\sigma}(R, A)$ *is an isomorphism.*

Proof of the claim. Recall from [AH] that if M and N are DG A-modules, then their differential graded Tor may be computed by the formula $\mathrm{Tor}^A(M,N) = H(F \otimes_A N)$, where F is a *semi-free resolution* of M, that is, a DG A-module F equipped with a quasi-isomorphism $F \to M$ and with a filtration $0 \subseteq F(0) \subseteq F(1) \subseteq \ldots$ such that $\cup_{i=0}^{\infty} F(i) = F$, and for each $i \geq 1$ the DG A-module $F(i)/F(i-1)$ is free on a basis of cycles.

Let F be a semi-free resolution of R as a DG module over the DG algebra $A \otimes A$. By the $(Q \otimes Q)$-flatness of P, the homomorphism $G = P \otimes_{Q\otimes Q} F \to P \otimes_{Q\otimes Q} R = R$ is a quasi-isomorphism of DG U-modules, nd together with the filtration $\{P \otimes_{Q\otimes Q} F(i)\}_{i\geq 0}$ this makes G into a semi-free resolution of R over U. Thus, the homomorphism $\operatorname{Tor}^{\sigma}(R, A)$ is the map induced in homology by the morphism of complexes

$$F \otimes_{A\otimes A} A \to G \otimes_U A$$

which sends $f \otimes a$ to $(1 \otimes f) \otimes a$. By the associativity of the tensor product, this is an isomorphism. □

In view of the preceding claims we obtain

$$\mathrm{HH}_*(R) = \operatorname{Tor}^{R\otimes R}(R, R) \cong \operatorname{Tor}^U(R, A).$$

Our last construction is that of a semi-free resolution V of the DG U-module $A = Q[X]$. We shall find V in the form $U\langle Y\rangle$, where $Y = \bigoplus_{n\geq 1} Y_n$ is a k-graded vector space, $\Gamma_k\langle Y\rangle$ is the free divided powers algebra on Y, and $U\langle Y\rangle = U \otimes \Gamma_k\langle Y\rangle$. For the technical details on DG algebras with divided powers the reader is referred to [GL]. Here we only note the formula

(5) $\partial(\gamma^j(y)) = \partial(y)\gamma^{j-1}(y)$ *where* $\gamma^j(y)$ *is the* j*'th divided power of* $y \in Y$.

A specific V is provided by the next result which has several antecedents in the framework of connected DG algebras over a field ([Ha, proof of (5.2)], [FHJLT, (1.9)], [HV, (4.7)]). We write sX for the graded vector space with $(sX)_i = X_{i-1}$ for $i \in \mathbb{Z}$. If x is an element of X_{i-1} then sx is its canonical image in $(sX)_i$. With all the notation introduced so far, we can now state the following theorem.

Theorem 3. *Let* Y_1 *be a* k*-vector space concentrated in degree one, with basis* $y_1, \ldots, y_n$*, and set* $Y = Y_1 \oplus sX$*.*

The homomorphism of DG algebras $\rho : U = P[X', X''] \to Q[X] = A$ *factors as a composition* $U \xrightarrow{\iota} U\langle Y\rangle \xrightarrow{\tau} A$*, where* ι *is the natural injection, and the following hold.*

(6) $\tau(y) = 0$ *for* $y \in Y$*, and* $\tau(\gamma^j(y)) = 0$ *for* $j \geq 2$ *and* $y \in Y$ *with* $|y|$ *even.*

(7) τ *is a quasi-isomorphism.*

(8.1) *if* $1 \leq i \leq n$*, then* $\partial y_i = u_i$*, where* $u_1, \ldots, u_n$ *is a minimal system of generators of the ideal* $\operatorname{Ker}(\mu : P \to Q)$*;*

(8.2) *if* $x \in X_q$ *with* $q \geq 1$*, and* $y = sx$*, then* $\partial(y) - x' - x'' \in (\mathfrak{m} + X'_{\leq q-1} + X''_{\leq q-1})V$*.*

The proof of Theorem 3, which relies on an extension of the arguments of the papers quoted above, will be given after the end of the proof of Theorem 2, to which we now return. The quasi-isomorphism $\tau : V \to A$ clearly yields a semi-free resolution of the DG U-module A; hence,

$$\operatorname{Tor}^U(R, A) \cong H(R \otimes_U V) = H(R\langle Y\rangle).$$

Let $\overline{\partial}$ denote the differential of the DG algebra $R\langle Y\rangle$. Noting that in $V = U\langle Y\rangle$ one has $u_i \in \operatorname{Ker}(\tau : V \to A) \subseteq \operatorname{Ker}(\pi\tau : V \to R)$, we see that $\overline{\partial} y_i = 0$ for $1 \leq i \leq n$. Thus, in particular, $z_i = y_1 \cdots y_i$ is a cycle of degree i in $R\langle Y\rangle$, for

$1 \le i \le n$. Furthermore, note that $X_1 \ne 0$ by (3), hence Y_2 contains an element $y_{n+1} = sx \ne 0$ for some $x \in X_1$. As $\partial y_{n+1} \in Y_1 = Ry_1 \oplus \cdots \oplus Ry_n$, formula (5) shows that $z_{n+j} = y_1 \cdots y_n \gamma^j(y_{n+1})$ is a cycle of degree $n+2j$ in $R\langle Y\rangle$.

In order to finish the proof of Theorem 2 it thus remains to show that z_i is not a boundary for $i \ge 1$. This follows from the fact that the z_i's are part of a homogeneous basis of the free R-module $R\langle Y\rangle$, hence cannot belong to $\partial(R\langle Y\rangle)$, which—by (5), (8.1), and (8.2) above—is contained in $\mathfrak{n}(R\langle Y\rangle)$. $\square$

Proof of Theorem 3. We first fix some notation and conventions. Choose a homogeneous basis $\{x_i\}_{i\ge n+1}$ of X, such that $|x_i| \le |x_j|$ for $n+1 \le i < j$. Setting $y_i = sx_i$ for $i \ge n+1$, we get a homogeneous basis $\{y_i\}_{i\ge 1}$ of Y, with the property that $|y_i| \le |y_j|$ for $1 \le i < j$. We make the convention that for an element of odd degree $y \in Y$, the symbol $\gamma^j(y)$ is defined only for $0 \le j \le 1$, by the equalities $\gamma^0(y) = 1$ and $\gamma^1(y) = y$. If $\mathbf{j} = (j_1, \dots, j_t, \dots)$ is a sequence of integers, with $j_t \ge 0$ and $j_t = 0$ for $t >> 0$, then we set $\gamma^{\mathbf{j}}(y) = \prod_{t\ge 1} \gamma^{j_t}(y_t)$, and note that these elements form a homogeneous basis of $\Gamma_k\langle Y\rangle$ over k.

To start the argument for the proof we note that condition (6) completely determines τ, hence the theorem will be proved once we show that the differential of U can be extended to one on $U \otimes \Gamma_k\langle Y\rangle$ in such a way that (7) and (8) hold. Let $\tau_{(q)}$ denote the homomorphism

$$V_{(q)} = P[X'_{\le q}, X''_{\le q}]\langle Y_{\le q+1}\rangle \to Q[X_{\le q}] = A_{(q)}$$

obtained from τ by restriction. As $\tau = \varinjlim \tau_{(q)}$, it suffices to show by induction on q that the differential of $V_{(q)}$ can be extended to a differential ∂ on $V_{(q+1)}$, for which conditions (8) are satisfied, and $\tau_{(q+1)}$ is a quasi-isomorphism.

For the basis of the induction, note that (8.1) uniquely determines the differential on $V_{(0)} = P\langle y_1, \dots, y_n\rangle$, which becomes the Koszul complex (over the ring P) on the elements $u_1, \dots, u_n$. Since by (4) they form a P-regular sequence, the homomorphism $H(\tau_{(0)}) : H(V_{(0)}) \to H(A_{(0)}) = Q$ is an isomorphism, as desired.

Next, let q be a non-negative integer, and assume that ∂ has been constructed on $V_{(q)}$ with the required properties. Let $x_{r+1}, \dots, x_s$ be the x_i's of degree $q+1$: they form a basis of X_{q+1}. Note that for $r+1 \le i \le s$ the elements $\partial x'_i - \partial x''_i$ are degree q cycles in $V_{(q)}$ and that they lie in $\mathrm{Ker}\tau_{(q)}$. Note also that $\tau_{(q)}$, which extends the surjection π, is itself surjective, and it is a quasi-isomorphism by the induction hypothesis. Thus, the homology exact sequence associated with

$$0 \to \mathrm{Ker}\ \tau_{(q)} \to V_{(q)} \to A_{(q)} \to 0$$

shows that $H(\mathrm{Ker}\ \tau_{(q)}) = 0$. It follows that there exist $v_i \in \mathrm{Ker}\ \tau_{(q)}$ such that $\partial(x'_i - x''_i) = \partial v_i$. We now set $\partial(y_i) = x'_i - x''_i - v_i$ for $r+1 \le i \le s$. In view of (5), this condition completely determines ∂ on $V_{(q+1)}$. Furthermore, as $v_i \in \mathrm{Ker}\ \tau_{(q)}$, a quick computation shows that $\tau_{(q+1)}$ becomes a homomorphism of DG algebras.

Consider in $U_{(q+1)} = P[X'_{\le q+1}, X''_{\le q+1}]$ the DG ideal $J = (\mathfrak{m} + X'_{\le q} + X''_{\le q})U_{(q+1)}$, and let $W_{(q+1)}$ denote the DG subalgebra $U_{(q+1)}\langle Y_{\le q+1}\rangle$ of $V_{(q+1)}$. We use the following isomorphisms of free modules over $B = k[X'_{q+1}, X''_{q+1}]$ as identifications:

$$W_{(q+1)}/JW_{(q+1)} \cong k\langle Y_{\le q+1}\rangle \otimes B;$$
$$JW_{(q+1)}/J^2W_{(q+1)} \cong (\mathfrak{m}/\mathfrak{m}^2 \oplus X'_{\le q} \oplus X''_{\le q}) \otimes k\langle Y_{\le q+1}\rangle \otimes B.$$

For $1 \leq i \leq n$, let x'_i be the image of u_i in $\mathfrak{m}/\mathfrak{m}^2$, they are linearly independent by (4). $\mathbf{j} = (j_1, \ldots, j_t, \ldots)$ as above, set $\mathbf{j}_t = (j_1, \ldots, j_t - 1, \ldots)$ if $j_t \geq 1$, and $\gamma^{\mathbf{j}_t}(y) = 0$ if $j_t = 0$. The non-zero ones among the elements $\{x'_t \otimes \gamma^{\mathbf{j}_t}(y)\}_{1 \leq t \leq r}$ and $\{x''_t \otimes \gamma^{\mathbf{j}_t}(y)\}_{n+1 \leq t \leq r}$ are then linearly independent; let $W_{\mathbf{j}}$ be their k-linear span in $JW_{(q+1)}/J^2W_{(q+1)}$.

By the inductive assumption, condition (8.2) holds for $y \in Y_{\leq q+1}$, hence $\partial W_{(q+1)} \subseteq JW_{(q+1)}$. On the other hand, by conditions (2.1) and (2.2), $\partial(J) \subseteq J^2$, hence $\partial(JW_{(q+1)}) \subseteq J^2W_{(q+1)}$. Thus, the connecting homomorphism of the homology exact sequence associated with

$$0 \to JW_{(q+1)}/J^2W_{(q+1)} \to W_{(q+1)}/J^2W_{(q+1)} \to W_{(q+1)}/JW_{(q+1)} \to 0$$

provides a B-linear homomorphism $\delta : W_{(q+1)}/JW_{(q+1)} \to JW_{(q+1)}/J^2W_{(q+1)}$ which maps a basis element $\gamma^{\mathbf{j}}(y)$ to a linear combination with coefficients ± 1 of the basis elements of $W_{\mathbf{j}}$. As the sum of the distinct subspaces $W_{\mathbf{j}}$ is direct, it follows that $\operatorname{Ker} \delta = k = (W_{(q+1)}/JW_{(q+1)})_0$. Let now z_i be the cycle $x'_i - x''_i - v_i$ for some i, $r+1 \leq i \leq s$. Using overbars to denote images in $W_{(q+1)}/JW_{(q+1)}$, we have

$$\delta(\overline{v}_i) = \delta\overline{x}'_i - \delta\overline{x}''_i - \delta\overline{z}_i = \delta\overline{x}'_i - \delta\overline{x}''_i = 0,$$

the last equality coming from the fact that $\partial x'_i$ and $\partial x''_i$ are in $J^2V_{(q)}$ by (2.1) and (2.2). It follows that $\overline{v}_i = 0$, hence $v_i \in (\mathfrak{m} + X'_{\leq q} + X''_{\leq q})V_{(q+1)}$ as desired.

To complete the inductive construction, we have to show that $H(\tau_{(q+1)})$ is an isomorhism. To this end, consider on $V_{(q+1)} = V_{(q)}[X'_{q+1}, X''_{q+1}]\langle Y_{q+2}\rangle$ an increasing filtration, whose p'th level is the $V_{(q)}$-linear span of the elements

$$(x'_{r+1})^{h'_{r+1}} \ldots (x'_s)^{h'_s}(x''_{r+1})^{h''_{r+1}} \ldots (x''_s)^{h''_s}\gamma^{j_{r+1}}(y_{r+1}) \ldots \gamma^{j_s}(y_s)$$

with $h'_i, h''_i, j_i \geq 0$ and $\sum_{i=r+1}^{s}(h'_i + h''_i + j_i) \leq p$. Similarly, consider on $A_{(q+1)} = A_{(q)}[X_{q+1}]$ the filtration whose p'th level is the $A_{(q)}$-linear span of the monomials $(x_{r+1})^{h_{r+1}} \ldots (x_s)^{h_s}$ with $h_i \geq 0$ and $\sum_{i=r+1}^{s} h_i \leq p$.

Clearly, the homomorphism $\tau_{(q+1)}$ preserves the filtrations and hence yields a homomorphism of the corresponding spectral sequences. At the E^o-level, the differential ∂^o on $E^o(V_{(q+1)})$ is equal to ∂ on $V_{(q)}$, and satisfies $\partial^o(x'_i) = 0 = \partial^o(x''_i)$ and $\partial^o(y_i) = z''_i \equiv x'_i - x''_i \mod (\mathfrak{m}X'_{q+1} + \mathfrak{m}X''_{q+1})$ for $r+1 \leq i \leq s$. On the other side, the differential ∂^o on $E^o(A_{(q+1)})$ is equal to ∂ on $A_{(q)}$, and satisfies $\partial^o(x_i) = 0$ for $r+1 \leq i \leq s$. Thus, at the E^1-level we get a homomorphism

$$E^1(\tau_{(q+1)}) : H(V_{(q)}) \otimes H(k[X'_{q+1}, Z''_{q+1}]\langle Y_{q+2}\rangle) \to H(A_{(q)}) \otimes k[X_{q+1}].$$

It is easy to see that $E^1(\tau_{(q+1)}) = H(\tau_{(q)}) \otimes H(\epsilon_{(q+1)})$, where $\epsilon_{(q+1)}$ is the homomorphism $k[X'_{q+1}, Z''_{q+1}]\langle Y_{q+2}\rangle \to k[X_{q+1}]$, which sends x'_i to x_i, and maps z''_i and $\gamma^j(y_i)$ $(j \geq 1)$ to 0. By the inductive assumption $H(\tau_{(q)})$ is an isomorphism, and a classical computation shows that $H(\epsilon_{(q+1)})$ is an isomorphism. Thus, $E^1(\tau_{(q+1)})$ is an isomorphism, and this implies $H(\tau_{(q+1)})$ is an isomorphism, as desired.

The inductive step in the construction of the differential ∂ on $V_{(q+1)}$ is now complete. This finishes the proof of Theorem 3, and with it the proofs of all the results of the paper. $\square$

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